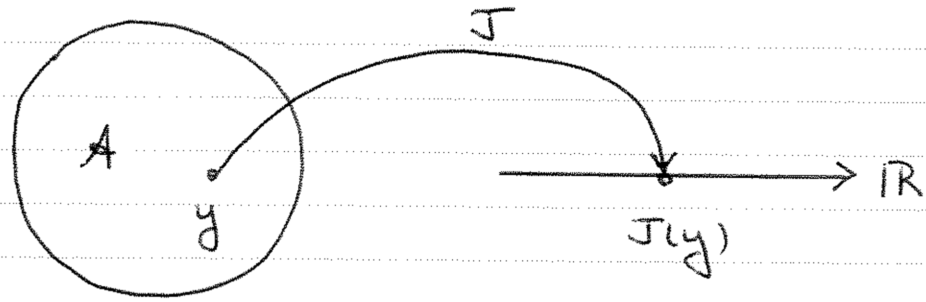
Functionals J

Defn: Let $\mathcal{A}$ be a set of functions $y : \mathbb{R}^m \to \mathbb{R}^n$. By a functional J on $\mathcal{A}$ we mean a rule that associates a unique real number $J(y)$ to each $y \in \mathcal{A}$



$$J : \mathcal{A} \to \mathbb{R}$$

Remarks

(1) A functional J is a type of function. Generally a function F associates a unique element $F(a) \in B$ to each $a \in A$ where $F : A \to B$ and A, B are any sets.

(2) The domain of J is $\mathcal{A}$

(3) The range $R(J)$ of J is some subset of $\mathbb{R}$, i.e., $R(J) \subset \mathbb{R}$.

Common Function Spaces

Let $y : [a,b] \to \mathbb{R}$ be real valued functions on the closed interval $[a,b]$

$C[a,b]$ = set of all functions $y(x)$ that are continuous on $[a,b]$

$C^n[a,b]$ = set of all functions $y(x)$ that are n-times continuously differentiable on $[a,b]$

A theorem from calculus states that if $y(x)$ is differentiable at x_0 it is also continuous at x_0. Hence if $y(x)$ has a continuous derivative $y'(x)$ on $[a,b]$ it is also continuous on $[a,b]$. In set notation the latter statement is

$$C^1[a,b] \subset C[a,b]$$

Moreover,

$$C^2[a,b] \subset C^1[a,b] \subset C[a,b]$$

<u>EXAMPLE</u> $A = \{ y : y \in C[0,\pi],\ y(0) = y(\pi) = 0 \}$

Then $y(x) = x \notin A$ but $y(x) = \sin x \in A$. Clearly then

$$A \subset C[0,\pi]$$

Examples of Functionals

EX $J : A \to \mathbb{R}$, $A = C[0,1]$, $J(y) \equiv y(\frac{1}{2})$

This is the point evaluation functional

$y(x) = x \qquad J(y) = \frac{1}{2}$

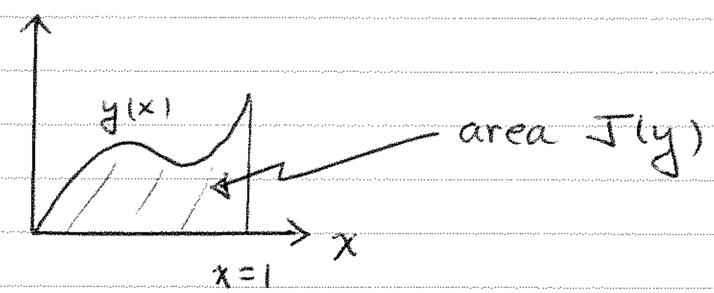
$y(x) = \sin(\pi x) \qquad J(y) = \sin(\frac{\pi}{2}) = 1$

EX Area functional

$$A = \{ y : y \in C[0,1],\ y(x) \geq 0 \}$$

$$J(y) = \int_0^1 y(x)\,dx$$

If $y \in A$, $J(y)$ is the area under the graph



Note $A \subset C[0,1]$ since not all $y \in C[0,1]$ are positive

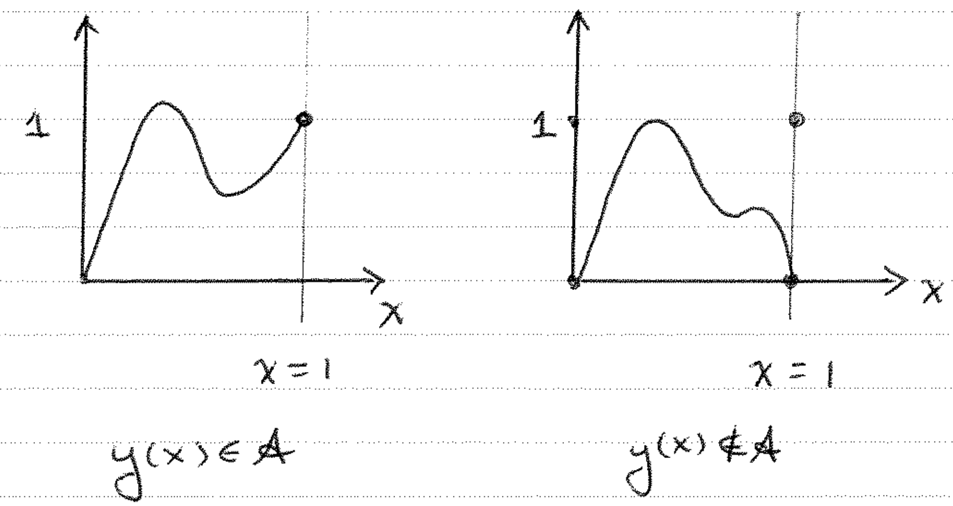
$y(x) = x^2 \qquad J(y) = \int_0^1 x^2\,dx = \frac{1}{3}$

Ex **Arclength functional**

$$A = \{ y : y \in C^1[0,1],\ y(0)=0,\ y(1)=1 \}$$

$$J(y) = \int_0^1 \sqrt{1 + y'(x)^2}\, dx$$

Here A is defined as those continuously differentiable $y(x)$ with $y(0)=0$ and $y(1)=1$.



So, for instance, $y(x) = x^{3/2} \in A$ and

$$J(y) = \int_0^1 \sqrt{1 + (\tfrac{3}{2}x^{1/2})^2}\, dx$$

$$J(y) = \int_0^1 \sqrt{1 + \frac{9x}{4}}\, dx = \frac{1}{27}(4+9x)^{3/2}\Big|_0^1$$

$$J(y) = \frac{1}{27}(13^{3/2} - 8)$$

Optimization of Functionals

A common and interesting problem associated with functionals is that of optimization. Specifically for a well defined class of functions $\mathcal{A}$ one may want to know the minimum or maximum value of $J(y)$, i.e.,

$$(1) \qquad \min_{y \in \mathcal{A}} J(y)$$

A common class of functionals associated with this issue have the form

$$(2) \quad J(y) = \int_a^b L(x, y(x), y'(x))\, dx$$

where, here,

$$L(x, y, y') = \text{Lagrangian.}$$

In general the answer to (1) will depend on the class of functions over which the functional J is being minimized

EX $\quad \mathcal{A} = \{ y : y \in C[a,b],\ y \geq 0 \}$

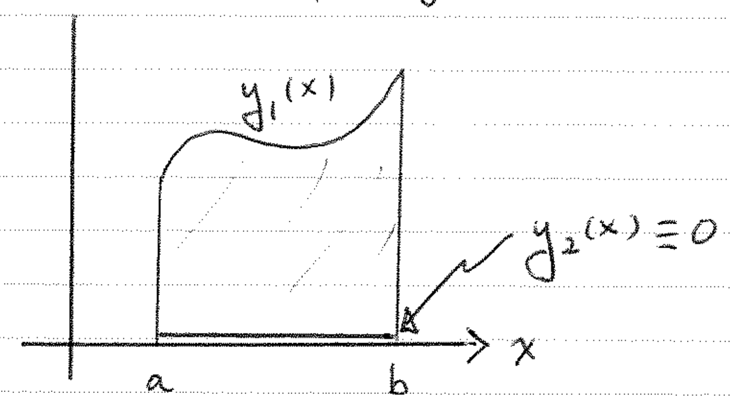
$$J(y) = \int_a^b y(x)\,dx \quad , \quad L(x,y,y') = y$$

Here $J(y)$ is the area under the curve defined by $y(x) \in \mathcal{A}$. Because of how $\mathcal{A}$ is defined, $y(x) \geq 0$ and $J(y) \geq 0$.

Clearly

$$\min_{y \in \mathcal{A}} J(y) = 0$$

and is attained by $y(x) \equiv 0$.



In the figure above $J(y_1) > 0$, $J(y_2) = 0$

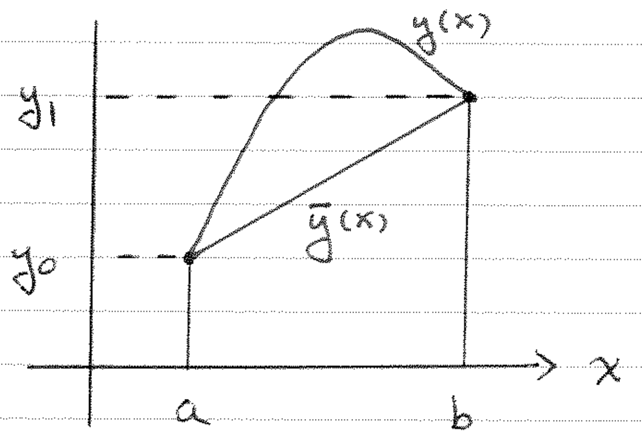
Note that $J(y)$ has no maximum value.

EX $\quad \mathcal{A} = \{ y : y \in C^1[a,b],\ y(a) = y_0,\ y(b) = y_1 \}$

$$J(y) = \int_a^b \sqrt{1 + (y'(x))^2}\, dx$$

where the Lagrangian $L(x, y, y') = \sqrt{1 + (y')^2}$
and

$$J(y) = \text{arclength of } y(x) \text{ on } x \in [a,b]$$



$J(y) \geq J(\bar{y})$

for all $y \in \mathcal{A}$

Clearly the minimum value is attained by the straight line from (a, y_0) to (b, y_1) or by

$$\bar{y}(x) = \frac{y_1 - y_0}{b - a}(x - a) + y_0$$

Again no maxima is attained and the minimum value is

$$J(\bar{y}) = \sqrt{1 + m^2}\,(b - a) \qquad m = \frac{y_1 - y_0}{b - a}$$

EX Answer depends on the admissible set $\mathcal{A}$

$$J(y) = \int_0^1 ((y')^2 - 1)^2 \, dx \geq 0$$

where here the Lagrangian $L(x,y,y') = ((y')^2-1)^2$.

Clearly the minimum value of $J(y)$ is zero if the integrand is zero

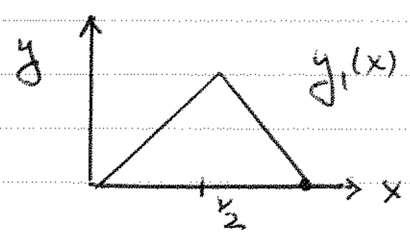
$$(y')^2 = 1 \qquad \forall x \in [0,1]$$

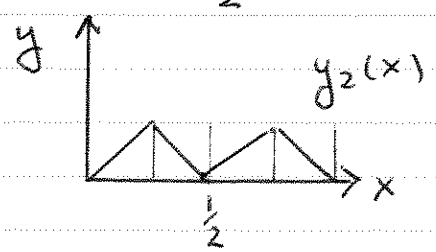
CASE ONE $\quad \mathcal{A} = C^1[0,1]$

$$y_\pm(x) = \pm x$$

are two functions which minimize J and $J(y_\pm) = 0$.

CASE TWO $\quad \mathcal{A}$ = set of piecewise differentiable functions on $[0,1]$ and $C[0,1]$



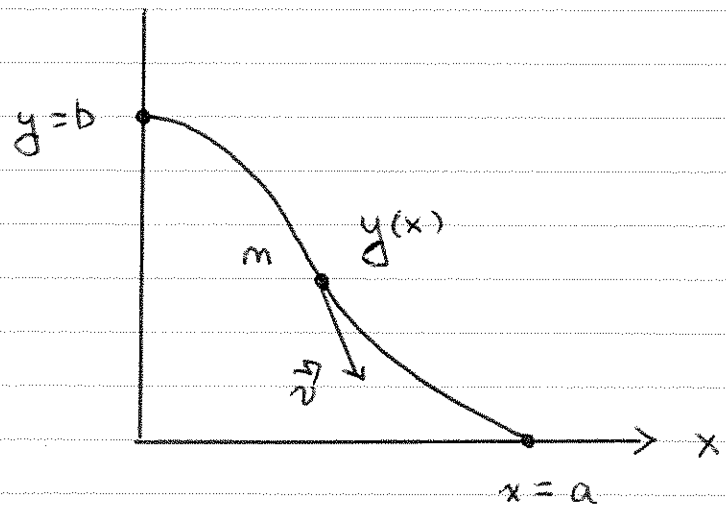


At the left are functions in $\mathcal{A}$ that have $y' = \pm 1$ on some interval. They are also continuous, and

$$J(y_1) = J(y_2) = 0$$

Clearly there are an infinite number of $y_K(x)$ for which $J(y_K) = 0$, $y_K \in \mathcal{A}$

Ex Brachistochrone Problem



Bernoulli (1696)

A bead of mass m slides down a (frictionless) wire under the influence of gravity. The bead is initially at rest at $(0, b)$.

Question: Of all curves $y(x)$ starting at $(0, b)$ and ending at $(a, 0)$, which minimizes the transit time?

Let

$$\vec{v} = \text{velocity}$$

$$v = \text{speed } |\vec{v}|$$

$$s = \text{arclength}$$

and ds be the arc length element

$$ds = \sqrt{1 + y'(x)^2}\, dx$$

For a given curve $y(x)$ of arclength L the total transit time T is

$$T = \int_0^T dt = \int_0^L \frac{dt}{ds}\, ds = \int_0^L \frac{1}{v}\, ds$$

and given $ds = \sqrt{1 + y'(x)^2}\, dx$ we get

$$(1) \qquad T = \int_0^a \frac{\sqrt{1 + y'(x)^2}}{v}\, dx$$

where v is the speed at position x. This is found using conservation of energy

$$(2) \qquad E = \underbrace{\frac{1}{2} m v^2}_{\text{kinetic}} + \underbrace{mgy}_{\text{potential}}$$

where g is the gravitational constant.

At time $t = 0$, $v = 0$ and $y = b$ by the problem constaint. Thus the constant E in (2) is $E = mgb$. Solving

$$\cancel{m}gb = \frac{1}{2}\cancel{m}v^2 + \cancel{m}g\, y(x)$$

for v and using the result in (2) we find:

$$(3) \quad T(y) = \int_0^a \frac{\sqrt{1+y'(x)^2}}{\sqrt{2g(b-y(x))}}\,dx$$

Defining the admissible set

$$A = \{ y : y \in C^1[0,a],\ y(0)=b,\ y(a)=0 \}$$

the minimization problem is succintly stated as

$$(4) \qquad \min_{y\in A} T(y)$$

Here the Lagrangian L is

$$L(x,y,y') = \sqrt{\frac{1+y'(x)^2}{2g(b-y(x))}}$$

The $\bar{y}(x)$ which solves (4) is not a straight line but rather a "cycloid". We shall see this later.

For a straight line $y_1 = b(1-x/a) \in A$ one can actually compute $T(y_1)$

$$T(y_1) = \sqrt{\frac{2(a^2+b^2)}{gb}} > T(\bar{y})$$