Natural Boundary Conditions - Motivation

In some instances boundary conditions on admissible functions may not be specified. For example, minimizing

$$ J(y) = \int_a^b L(x, y, y') \, dx $$

over the admissible set

$$ A = \{ y \in C^2[a, b] : y(a) = 0 \} $$

Here \( y(b) \) is not known and part of the problem is to find it. Before we develop theory to find such a "natural boundary condition" we present a real world problem:

What path should a boat take to minimize the transit time \( T \) across a river whose velocity is \( \vec{v} = v(x) \hat{j} \)?

Here \( T = T(y) \) depends on \( y(x) \) hence is a functional and the endpoint \( y(b) \) is not known.
Steer problem functional derivation

Assume that if there were no flow the speed of the boat is constant. This would correspond to a constant throttle position say.

Without current

\[
\begin{align*}
\text{Position at time } t &= (X(t), Y(t)) \\
C &= \sqrt{\dot{X}^2 + \dot{Y}^2}
\end{align*}
\]

With current

\[
\begin{align*}
\text{Position at time } t &= (X(t), Y(t)) \\
\text{Current Velocity} &= u(x) \hat{j}\\
\end{align*}
\]

Since the current flows in y-direction it only affects the \(j\) component of the boat velocity

\[
\begin{align*}
\dot{x} &= \ddot{X} \\
\dot{j} &= \ddot{Y} + u
\end{align*}
\]
Suppose the boat path is \( y = y(x) \).

Then

\[
\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}
\]

Notationally we let \( y'(x) = \frac{dy}{dx} \) so that

\[
(3) \quad \dot{x} = \dot{x} \quad y'(x) \dot{x} = \ddot{y} + v
\]

We use (3) in (1) to determine \( \dot{x} \)

\[
c^2 = \dot{x}^2 + \ddot{y}^2
\]

\[
c^2 = \dot{x}^2 + (y' \dot{x} - v)^2
\]

Expanding this out yields a quadratic equation for \( \dot{x} \)

\[
(1 + y'^2) \dot{x}^2 - 2vy' \dot{x} + (v^2 - c^2) = 0
\]

Solving for \( \dot{x} \) with quadratic formula

\[
(4) \quad \dot{x} = \frac{dx}{dt} = \frac{vy' \pm \sqrt{y'^2 - (1+y'^2)(v^2-c^2)}}{(1+y'^2)}
\]

We must have \( c > v \) else the boat could not cross. With this condition we must take + in (4) so that \( \dot{x} > 0 \) and the boat crosses.
Transit time $T$

\[ T = \int_{P}^{Q} dt = \int_{0}^{b} \frac{dt}{dx} dx \]

where $\frac{dt}{dx}$ is the reciprocal of $\frac{dx}{dt}$ in eqn (4).

After some algebraic simplification,

\[ T(y) = \int_{0}^{b} \left( \frac{dx}{dt} \right)^{-1} dx = \int_{0}^{b} L(x, y, y') dx \]

where

\[ L = \frac{\sqrt{c^2(1+y'^2) - \nu^2}}{c^2 - \nu^2} \]

and the admissible set is

\[ A = \{ y \in C^2[0, b] : y(0) = 0 \} \]
Natural Boundary Conditions

Define

\[ J(y) = \int_a^b L(x, y, y') \, dx \]

where the admissible set is

\[ A = C^2[a, b] \quad \text{(admissible set)} \]

Here no boundary conditions for \( y \in A \) have been stipulated. As a consequence the set of all admissible variations has none either

\[ A^* = C^2[a, b] \quad \text{(admissible variations)} \]

As before we let \( \bar{y} \in A \) be an extrema and set

\[ F(\varepsilon) = J(\bar{y} + \varepsilon h) \]

\[ F(\varepsilon) = \int_a^b L(x, \bar{y} + \varepsilon h, \bar{y}' + \varepsilon h') \, dx \]

As before

\[ F'(\varepsilon) = \int_a^b L_y h + L_{y'} h' \, dx \]

Integrate by parts and evaluate at \( \varepsilon = 0 \) to determine the first variation.

\[ F'(0) = \delta J(\bar{y}, h) \]
\[ S\bar{J} = \int_{a}^{b} \left( L_y - \frac{d}{dx} L_y \right) h(x) \, dx \]

\[ \uparrow \quad \text{boundary terms} \quad \uparrow \quad \text{integral} \]

If \( \bar{y} \in A \) is an extrema of \( J \), its first variation must vanish for all \( h \in A^* \).

This means each of the boundary and integral terms must vanish independently.

If we consider only those \( h \in A^* \) that vanish at the endpoints, then

\[ S\bar{J} = \int_{a}^{b} \left( L_y - \frac{d}{dx} L_y \right) h(x) \, dx \]

\[ \uparrow \quad \text{all } h \in A^* \text{ with } h(a) = h(b) = 0 \]

This can vanish only if

\[ (1) \quad L_y - \frac{d}{dx} L_y = 0 \quad \text{EL eqn.} \]

Thus, if \( \bar{y} \) solves (1)

\[ S\bar{J} = \left[ L_y'(x, \bar{y}(x), \bar{y}'(x)) h(x) \right]_{x=a}^{x=b} \]

This must vanish for all \( h \in A^* \), including those that do not vanish at \( x = a, b \).
Hence extrema must not only be solns of the EL-eqns but they must also satisfy the B.C.

\[
\begin{align*}
(2) & \quad L_y(x, y(a), y'(a)) = 0 \quad \{ \text{Natural} \} \\
(3) & \quad L_y'(b, y(b), y'(b)) = 0 \quad \{ \text{Boundary} \} \\
\end{align*}
\]

Collectively (1)-(3) form a BVP for extrema \( y(x) \).

**Example (Steering Problem)**

\[
L(x, y') = (c^2 - v^2)^{-1} \left( \Delta^{1/2} - vy' \right)
\]

where \( \Delta = c^2(1+y'^2) - v^2 \), \( v = v(x) \) and \( c \) constant.

Eqn (3) above is natural B.C. for \( x = b \).

\[
L_y' = (c^2 - v^2)^{-1} \left( \frac{c^2 y'}{\Delta^{1/2}} - v^2 \right) = 0
\]

The term in parentheses must vanish \( \Rightarrow \)

\[
\begin{align*}
& c^2 y' = v^2 \Delta^{1/2} \\
& c^4 y'' = v^2 (c^2(1+y'^2) - v^2)
\end{align*}
\]

Simplify, solve for \( y' \) and evaluate at \( x = b \)

\[
y'(b) = \frac{v(b)}{c} \quad \text{Natural B.C.}
\]

Slope at opposite bank should be that given in Nat. B.C.
EXAMPLE Define \( J: A \to \mathbb{R} \) by

\[
J(y) = y(1)^2 + \int_0^1 y'(x)^2 \, dx
\]

\[
A = \{ y \in C^2[0, 1] : y(0) = 1 \}
\]

Here the boundary condition at \( x = 1 \) is "free." The Lagrangian \( L = (y')^2 \) and the set of admissible variations is

\[
A^* = \{ h \in C^2[0, 1] : h(0) = 0 \}
\]

Seek to derive natural B.C. and extrema

\[
F(\varepsilon) = J(y_0 + \varepsilon h) = (y_0(1) + \varepsilon h(1))^2 + \int_0^1 L(y_0' + \varepsilon h') \, dx
\]

Compute \( F'(\varepsilon) \)

\[
F'(\varepsilon) = 2(y_0(1) + \varepsilon h(1)) h(1) + \int_0^1 L_y(y_0' + \varepsilon h') h' \, dx
\]

Evaluate at \( \varepsilon = 0 \) to get first variation.

\[
SJ = 2y_0(1) h(1) + \int_0^1 L_y(y_0') h' \, dx
\]

Integrate by parts (since \( h \in A^* \))

\[
SJ = (2y_0(1) + L_y(y_0'(1))) h(1) - L_y(y_0'(0)) h(0) - \int_0^1 \frac{d}{dx} L_y(y_0') h(x) \, dx
\]
Thus the first variation is

\[ S J = \left( \frac{d \bar{y}(x) + L_y'(\bar{y}(x))}{h(x)} \right) h(x) - \int_0^1 \frac{d}{dx} L_y'(\bar{y}') h(x) \, dx \]

= 0 for NBC

= 0 is EL-eqn

Since \( L_y' = 2y' \) we conclude extrema must satisfy

1. \( \frac{d}{dx} L_y' = 0 \) EL-eqn.

2. \( \bar{y}(0) = 1 \) given B.C.

3. \( \bar{y}(1) + \bar{y}'(1) = 0 \) N.B.C.

Explicitly eqn (1) is

\[ \frac{d}{dx} (2\bar{y}') = 2\bar{y}'' = 0 \]

whose general solution is \( \bar{y}(x) = Ax + B; A, B \in \mathbb{R} \).

\[ \bar{y}'(x) = Ax, \quad \bar{y}(x) = Ax + B \]

Thus, B.Conds are

\[ \bar{y}(0) = B = 1 \]

\[ \bar{y}(1) + \bar{y}'(1) = aA + B = 0 \]

whose soln is \( A = -\frac{1}{2}, B = 1 \) and the extrema is

\[ \bar{y}(x) = -\frac{1}{2}x + 1 \]