Dimensional Analysis

q = physical quantity

$[q]$ = units of q

The units of q depend on the unit system used.

v = velocity

$[v]$ = m/sec MKSA

$[v]$ = ft/sec Imperial

Unit systems typically have a set of fundamental dimensions which can be used to represent the units of q.

Given the fundamental dimensions

$$L_1, L_2, \dots L_n$$

it is assumed the units of a physical quantity can be written

$$[q] = L_1^{a_1} L_2^{a_2} \cdots L_n^{a_n}$$

for certain exponents $a_1, a_2 \dots a_n$

Example Fundamental MKSA dimensions

q_1 = time $[q_1] = L_1 = T$ sec

q_2 = length $[q_2] = L_2 = L$ m (meter)

q_3 = mass $[q_3] = L_3 = M$ kg

q_4 = current $[q_4] = L_4 = A$ amp

q_5 = temperature $[q_5] = L_5 = K$ °K (kelvin)

Example The MKSA units of force F and energy E are Newtons N and Joules J, respectively.

But using $F = ma$, $E = \frac{1}{2} m v^2$

$$[F] = [m][a] = M L T^{-2}$$

$$[E] = [m][v]^2 = M L^2 T^{-2}$$

Using the established notation conventions with the fundamental MKSA dimensions defined above

$$[q] = T^{a_1} L^{a_2} M^{a_3} A^{a_4} K^{a_5}$$

For force

$$(a_1, a_2, a_3, a_4, a_5) = (-2, 1, 1, 0, 0)$$

For energy

$$(a_1, a_2, a_3, a_4, a_5) = (-2, 2, 1, 0, 0)$$

Example Use the following physical laws to determine the units of voltage V in terms of the fundamental MKSA dimensions

(1) $\vec{F} = q\vec{E}$ force exerted on charge q in electric field $\vec{E}$

(2) $V = \int_C \vec{E}\cdot d\vec{r}$ change in potential (volt) between curve end points

From eqn (1) we have

$$[\vec{E}] = [\vec{F}][q]^{-1} = (MLT^{-2})(AT)^{-1}$$

$$[\vec{E}] = T^{-3} L\, M\, A^{-1}$$

Then from equation (2)

$$[V] = [\vec{E}][d\vec{r}]$$

(3) $$[V] = T^{-3} L^2 M A^{-1}$$

In physics V is often viewed as work per unit charge, i.e.

$$[V] = \frac{J}{Q} = \frac{\text{Joules}}{\text{Coulomb}}$$

One can verify $[JQ^{-1}] = T^{-3}L^2MA^{-1}$ as well.

Physical Law (Algebraic)

Is an assumed (algebraic) relation between certain physical quantities.

$$f(q_1, \ldots q_m) = 0$$

For example

x = falling body position

t = time

g = gravitational constant

then

$$x - \frac{1}{2} g t^2 = 0$$

in the absense of friction.

Dimensionless Parameters

For certain exponents α_j

$$\pi = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_m^{\alpha_m}$$

the parameter π has no units.

For the example above

$$\pi = x g^{-1} t^{-2}$$

and the dimensionless form of the law is

$$\pi - \frac{1}{2} = 0$$

Buckingham Π - Theorem (informal)

For a physical law

(1) $\quad f(q_1, \dots q_m) = 0$

there is a set of fewer dimensionless parameters $\pi_1, \pi_2 \dots \pi_K$ and a function F such that

(2) $\quad F(\pi_1, \dots \pi_K) = 0 \qquad K < m$

is equivalent to (1).

Illustrate use by way of examples

EXAMPLE Falling body

Suppose we assume that the position x of a falling body depends only on time t and g, i.e.,

$$f(x, t, g) = 0$$

for some f.

$$\pi = x^{\alpha_1} t^{\alpha_2} g^{\alpha_3} = L^{\alpha_1 + \alpha_3} T^{\alpha_2 - 2\alpha_3}$$

is dimensionless only if

$$\alpha_1 + \alpha_3 = 0 \qquad \alpha_2 - 2\alpha_3 = 0$$

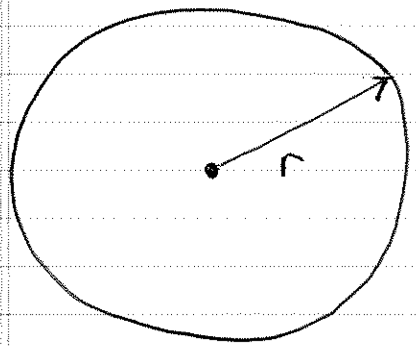
Thus, $\alpha_1 = 1$, $\alpha_2 = -2$, $\alpha_3 = -1$ and $\pi = \frac{x}{gt^2}$

$$F(\pi) = 0$$

so π is a root c of F and

$$x = cgt^2$$

EXAMPLE Atomic Bomb Wavefront



E = energy released

r = radius of shockwave at time t

ρ = initial air density

We shall assume these quantities are related

$$f(t, r, \rho, E) = 0$$

and seek the associated dimensionless law

$$F(\Pi_1, \ldots \Pi_k) = 0$$

STEP ONE (Find dimensionless Π)

$$[t] = T$$

$$[r] = L$$

$$[\rho] = M L^{-3}$$

$$[E] = M L^2 T^{-2}$$

(the latter from work is force × distance)

Let

$$\Pi = t^{\alpha_1} r^{\alpha_2} \rho^{\alpha_3} E^{\alpha_4}$$

where Π is dimensionless

$$[\Pi] = T^{\alpha_1} L^{\alpha_2} (ML^{-3})^{\alpha_3} (ML^2T^{-2})^{\alpha_4}$$

$$[\Pi] = T^{\alpha_1 - 2\alpha_4} L^{\alpha_2 - 3\alpha_3 + 2\alpha_4} M^{\alpha_3 + \alpha_4}$$

coefficients must vanish for Π to have no dimensions

Conclude Π is dimensionless if and only if

$$\boxed{\begin{aligned} \alpha_1 - 2\alpha_4 &= 0 \\ \alpha_2 - 3\alpha_3 + 2\alpha_4 &= 0 \\ \alpha_3 + \alpha_4 &= 0 \end{aligned}}$$

Linear system of 3 eqns for 4 unknowns.

Solve for α_K in terms of α_4

$$\alpha_1 = 2\alpha_4 \qquad \alpha_2 = -5\alpha_4 \qquad \alpha_3 = -\alpha_4$$

concluding all such Π have the form

$$\Pi = (t^2 r^{-5} \rho^{-1} E)^{\alpha_4}$$

wlog, pick $\alpha_4 = -1$ (to correspond to text)

$$\boxed{\Pi = \frac{r^5 \rho}{t^2 E}}$$

STEP TWO   Make conclusions

By the π - Theorem any relation

$$f(t, r, \rho, E) = 0$$

is equivalent to

$$F(\pi) = 0 \qquad \pi = \frac{r^5 \rho}{t^2 E}$$

for some F.

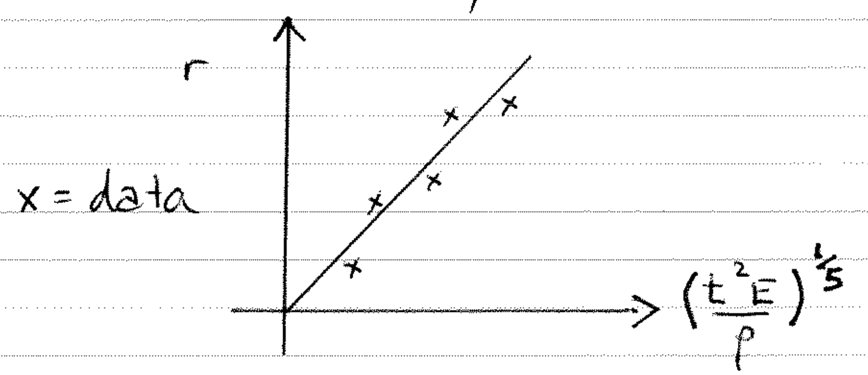
Since π is a root of F

$$\frac{r^5 \rho}{t^2 E} = \text{constant}$$

Solving for r, there is some constant c such that

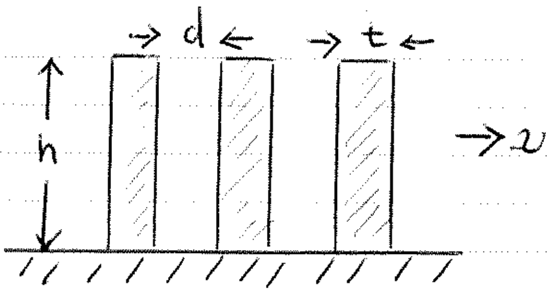
$$r = c \left(\frac{t^2 E}{\rho} \right)^{1/5}$$

For fixed E, ρ the $t^{2/5}$ dependence of blast diameter r on t was experimentally observed



determine slope c from data.

EXAMPLE Domino Problem



Dominos of thickness t spaced a distance d apart attain a terminal velocity v.

Depends on gravity hence gravitatonal constant g.

Special Case $t = d$

Assume quantities d, h, v, g are related

$$(1) \qquad f(d, h, v, g) = 0$$

and seek associated dimensionless law

$$F(\pi_1, \pi_2, \ldots \pi_K) = 0$$

STEP ONE Find dimensionless π

$$[d] = [h] = L$$

$$[v] = LT^{-1}$$

$$[g] = LT^{-2}$$

Let

$$\pi = d^{\alpha_1} h^{\alpha_2} v^{\alpha_3} g^{\alpha_4}$$

where π is dimensionless.

$$[\pi] = L^{(\alpha_1+\alpha_2+\alpha_3+\alpha_4)} \; T^{-(\alpha_3+2\alpha_4)}$$

must vanish for π to be dimensionless

Conclude π dimensionless if and only if

$$(2) \quad \boxed{\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= 0 \\ \alpha_3 + 2\alpha_4 &= 0 \end{aligned}}$$

Linear system of 2 eqns for 4 unknowns.

Mathematically if one defines

$$\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)^T \qquad A = \begin{array}{c} \quad\ \text{free} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \end{array} \in \mathbb{R}^{2\times 4}$$

the solutions of (2) are all those $\vec{\alpha}$ such that $A\vec{\alpha} = 0$. In other words $\vec{\alpha}$ is in the nullspace $N(A)$ of A.

$(\alpha_2, \alpha_4) = (0, 1)$ yields

$$\vec{\alpha} = (1, 0, -2, 1)$$

$$\pi_1 = dg/v^2$$

$(\alpha_2, \alpha_4) = (1, 0)$ yields

$$\vec{\alpha} = (1, -1, 0, 0)$$

$$\pi_2 = d/h$$

Two dimensionless quantities.

STEP TWO Make conclusions

By the π - Theorem there is some function F such that

$$F(\pi_1, \pi_2) = 0 \tag{3}$$

is equivalent to the original assumed law.

Solving (3) for π_1 there is some function ϕ such that

$$\pi_1 = \phi(\pi_2)$$

$$\frac{dg}{v^2} = \phi\left(\frac{d}{h}\right)$$

or equivalently some function Φ such that

$$\boxed{v = \sqrt{gd}\;\Phi\left(\frac{d}{h}\right)}$$

By measuring v for dominos of fixed width but varying heights one can curve fit Φ!

Case $t \neq d$ 5 quantities, 2 basic units

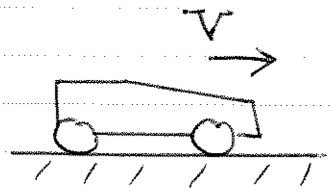
yields three dimensionless π_K

$$\pi_1 = \frac{v^2}{dg} \qquad \pi_2 = \frac{d}{t} \qquad \pi_3 = \frac{d}{h}$$

and ultimately

$$v = \sqrt{gd}\;\Phi\left(\frac{d}{h}, \frac{d}{t}\right)$$

EXAMPLE   Fuel economy tests for cars



Cars are driven at constant speed V.

Force of propulsion F is at equilibrium with air resistance, friction, etc.

Assume that F and V are related to

C = rate fuel being burned (vol/time)

K = energy/volume of fuel

Assume there exists a law such that

$$f(F, V, C, K) = 0$$

for some function f.

Find dimensionless parameters

$$[F] = MLT^{-2}$$
$$[V] = LT^{-1}$$
$$[C] = L^3T^{-1}$$
$$[K] = ML^{-1}T^{-2}$$

Let

$$\pi = F^{\alpha_1} V^{\alpha_2} C^{\alpha_3} K^{\alpha_4}$$

then

$$[\pi] = (MLT^{-2})^{\alpha_1}(LT^{-1})^{\alpha_2}(L^3T^{-1})^{\alpha_3}(ML^{-1}T^{-2})^{\alpha_4}$$

$$[\pi] = M^{(\alpha_1+\alpha_4)} L^{\alpha_1+\alpha_2+3\alpha_3-\alpha_4} T^{-2\alpha_1-\alpha_2-\alpha_3-2\alpha_4}$$

↑ ↑ ↑

must vanish

π is dimensionless if and only if

$$\begin{aligned} \alpha_1 \qquad\qquad\qquad\qquad &+ \alpha_4 = 0 \\ \alpha_1 + \alpha_2 + 3\alpha_3 &- \alpha_4 = 0 \\ 2\alpha_1 + \alpha_2 + \alpha_3 &+ 2\alpha_4 = 0 \end{aligned}$$

Soln(s) not immediately obvious so we row reduce the coefficient matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 3 & -1 \\ 2 & 1 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & -2 & \boxed{2} \end{bmatrix}$$

sole free variable

Having identified α_4 as the sole free variable, set $\alpha_4 = 1$ and back solve

$$\alpha_1 = -1 \qquad \alpha_2 = -1 \qquad \alpha_3 = 1 \qquad \alpha_4 = 1$$

yields one dimensionless quantity

$$\pi = \frac{CK}{FV}$$

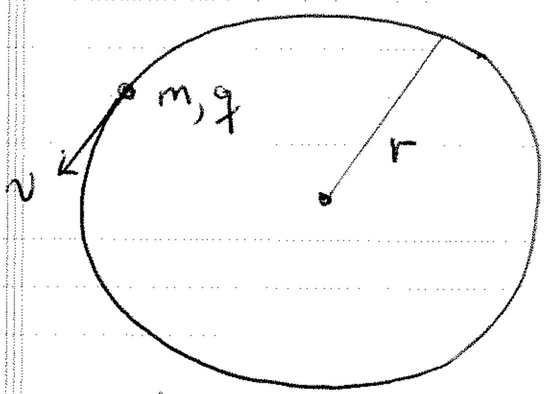
By the π-Theorem this must be constant from which we conclude

$$F = \alpha \left(\frac{CK}{V} \right)$$

where α is some (dimensionless) constant.

EXAMPLE Cyclotron

A charged particle of charge q and mass m enters transverse to a uniform magnetic field of magnitude B. The particle travels in a circle of radius r at speed v.



Field B.

Units of magnetism

$$\vec{F} = q \vec{v} \times \vec{B}$$

Force $\vec{B}$ exerts on charge q at velocity $\vec{v}$

$[B]$ = Tesla (MKSA)

$$[B] = M T^{-1} Q^{-1}$$

Units

$$[q] = Q \qquad [m] = M \qquad [r] = L$$

$$[v] = L T^{-1} \qquad [B] = M T^{-1} Q^{-1}$$

Dimensionless Parameters

$$\Pi = q^{\alpha_1} B^{\alpha_2} v^{\alpha_3} m^{\alpha_4} r^{\alpha_5}$$

$$[\Pi] = \underset{\uparrow}{Q^{\alpha_1 - \alpha_2}} \; \underset{\uparrow}{M^{\alpha_2 + \alpha_4}} \; \underset{\uparrow}{L^{\alpha_3 + \alpha_5}} \; \underset{\uparrow}{T^{-\alpha_2 - \alpha_3}}$$

vanish

Resulting system

$$\begin{array}{lllllll} \alpha_1 & - \alpha_2 & & & & = 0 \\ & \alpha_2 & & + \alpha_4 & & = 0 \\ & & \alpha_3 & & + \alpha_5 & = 0 \\ & \alpha_2 & + \alpha_3 & & & = 0 \end{array}$$

By inspection coefficient matrix has one dimensional nullspace

Set $\alpha_1 = 1$ and solve for other α_j

$$\vec{\alpha} = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (1, 1, -1, -1, 1)$$

$$\pi = \frac{qBr}{mv}$$

sole dimensionless quantity hence is constant.

Results in a law

$$(1) \quad \frac{mv}{r} = c\, qB \qquad c \in \mathbb{R}$$

From physics centripetal and magnetic forces must cancel.

$$(2) \quad \frac{mv^2}{r} = qvB$$

$$\text{centripetal} = \text{magnetic}$$

The "law" derived using dimensional analysis closely matches that in physics!