Unit Free Physical Laws

Each unit system $\mathcal{S}$ has its own fundamental dimensions $L_1, \dots L_n$. Conversion factors λ_i relate the units

$$\bar{L}_i = \lambda_i L_i$$

For example

$$L_1 = m \qquad L_2 = kg \qquad L_3 = sec \qquad (MKSA)$$

$$\bar{L}_1 = cm \qquad \bar{L}_2 = gm \qquad \bar{L}_3 = sec \qquad (CGS)$$

Physical quantities are then related

$$\bar{q} = \bar{L}_1^{\alpha_1} \cdots \bar{L}_n^{\alpha_n}$$

$$\bar{q} = (\lambda_1^{\alpha_1} \cdots \lambda_n^{\alpha_n})\, q$$

One expects physical laws to be independent of the units chosen thus motivating the following defn:

Definition

The physical law

$$f(q_1, \dots q_m) = 0$$

is said to be unit free iff

$$f(\bar{q}_1, \dots \bar{q}_m) = 0 \qquad \forall \lambda_i > 0$$

where $\bar{q}_k = (\lambda_1^{\alpha_{k1}} \cdots \lambda_n^{\alpha_{kn}})\, q$.

EXAMPLE   Unit free law

$$f(x,t,g) = x - \tfrac{1}{2} g t^2 = 0$$

Let $(\bar{x}, \bar{t}, \bar{g})$ be some other unit system

$$(1) \quad \bar{x} = \lambda_1 x$$

$$(2) \quad \bar{t} = \lambda_2 t$$

Since $[g] = L T^{-2}$ we have

$$\bar{g} = [\bar{x}][\bar{t}]^{-2}$$

$$(3) \quad \bar{g} = \lambda_1 \lambda_2^{-2} g$$

Using (1)-(3)

$$\begin{aligned} f(\bar{x}, \bar{t}, \bar{g}) &= \bar{x} - \tfrac{1}{2} \bar{g} \bar{t}^2 \\ &= \lambda_1 x - \tfrac{1}{2} \lambda_1 \cancel{\lambda_2^{-2}} g \cdot \cancel{\lambda_2^2} t^2 \\ &= \lambda_1 (x - \tfrac{1}{2} g t^2) \end{aligned}$$

$$\boxed{f(\bar{x}, \bar{t}, \bar{g}) = \lambda_1 f(x,t,g)}$$

Thus $f(x,t,g) = 0 \iff f(\bar{x}, \bar{t}, \bar{g}) = 0$.

As a specific example

x	(cm)	$\bar{x}$	(in)
t	(sec)	$\bar{t}$	(min)
g	(cm/sec^2)	$\bar{g}$	(in/min^2)

where $\lambda_1 \simeq \frac{1}{2.54} \frac{\text{in}}{\text{cm}}$ and $\lambda_2 = \frac{1}{60} \frac{\text{min}}{\text{sec}}$

Dimension Matrix and Dimensionless Quantities

$L_1, L_2, \dots L_n$ fundamental dimens.

$q_1, q_2, \dots q_m$ physical quantities

Given L_K we seek to find all dimensionless quantities

$$\pi = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_m^{\alpha_m}$$

where $\vec{\alpha} = (\alpha_1, \alpha_2, \dots \alpha_m)$. Given each physical quantity q_j has units

$$[q_j] = L_1^{a_{1j}} L_2^{a_{2j}} \cdots L_n^{a_{nj}}$$

then

$$[\pi] = (L_1^{a_{11}} L_2^{a_{21}} \cdots L_n^{a_{n1}})^{\alpha_1} \cdots (L_1^{a_{1m}} L_2^{a_{2m}} \cdots L_n^{a_{nm}})^{\alpha_m}$$

Expand and collect powers of L_K

$$[\pi] = L_1^{(a_{11}\alpha_1 + a_{12}\alpha_2 + \cdots a_{1m}\alpha_m)} \cdots L_n^{(a_{n1}\alpha_1 + \cdots a_{nm}\alpha_m)}$$

The requirement $[\pi] = 1$ leads to the system

$$\begin{aligned}
a_{11}\alpha_1 + a_{12}\alpha_2 + \cdots \quad a_{1m}\alpha_m &= 0 \\
a_{21}\alpha_1 + a_{22}\alpha_2 + \cdots \quad a_{2m}\alpha_m &= 0 \\
\cdot \qquad \cdot \qquad \cdot \qquad &= 0 \\
a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots \quad a_{nm}\alpha_m &= 0
\end{aligned}$$

Define the dimension matrix

$$A \equiv \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \in \mathbb{R}^{n \times m}$$

where number of rows = number of fundamental dimensions.

<u>Theorem</u> Given the fundamental dimensions $L_1, \ldots L_n$ the quantity

$$\pi = q_1^{\alpha_1} \cdots q_m^{\alpha_m}$$

is dimensionless if and only if $\vec{\alpha} = (\alpha_1, \ldots \alpha_m) \in N(A)$ nullspace of A. The dimension of the solution space is, therefore,

$$m - \operatorname{rank}(A)$$

where $\operatorname{rank}(A) = \dim \operatorname{col}(A) = \dim \operatorname{row}(A)$.

Pf/ trivial since $[\pi] = 1 \Leftrightarrow A\vec{\alpha} = \vec{0}$. Dimensionality follows from the fundamental theorem of linear algebra. $\square$

<u>Note</u>: Although there are an infinite number of π we, without loss of generality say there are $m - \operatorname{rank}(A)$.

EXAMPLE Falling body revisited

$$\pi = x^{\alpha_1} t^{\alpha_2} g^{\alpha_3}$$

Here there are $m = 3$ physical quantities and $n = 2$ fundamental dimensions

$$L_1 = L \qquad L_2 = T$$

Since

$$[\pi] = L^{\alpha_1 + \alpha_3} T^{\alpha_2 - 2\alpha_3}$$

the dimension matrix is

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$\uparrow$ $[g] = L T^{-2}$

Here the rows of A are independent (already upper echelon) hence rank(A) = 2

"Number" of dimensionless π $= m - \text{rank}(A) = 1$

Any basis vector $\vec{\alpha}$ for N(A) yields π

$$\vec{\alpha} = (-1, 2, 1)$$

yields

$$\pi = \frac{g t^2}{x}$$

Π - Theorem

Let $r = \text{rank}(A)$ and

$$(1) \qquad f(q_1, q_2, \ldots q_m) = 0$$

be a unit free law. Then there exist dimensionless $\Pi_1, \ldots \Pi_{m-r}$ and a function F such that

$$(2) \qquad F(\Pi_1, \ldots \Pi_{m-r}) = 0$$

if and only if (1) is satisfied.

Pf outline (details in text)

$$\Pi_k = q_1^{\alpha_1^{(k)}} q_2^{\alpha_2^{(k)}} \cdots q_m^{\alpha_m^{(k)}}$$

where $\vec{\alpha}_k = (\alpha_1^{(k)}, \ldots \alpha_m^{(k)})$ are basis vectors for $N(A)$. By permuting q_i one can wlog choose certain pairs of $\alpha_m^{(k)}$ equal to 0 and 1, such as

$$\Pi_1 = q_1^{\alpha_1^{(1)}} q_2^{\alpha_2^{(1)}} q_3^{0} q_4^{1} \qquad m = 4$$
$$\Pi_2 = q_1^{\alpha_1^{(2)}} q_2^{\alpha_2^{(2)}} q_3^{1} q_4^{0} \qquad n = 2$$

so that (q_3, q_4) can be expressed as functions of Π_1, Π_2, q_1, q_2.

Then define

$$G(q_1, q_2, \Pi_1, \Pi_2) \equiv f(q_1, q_2, \Pi_2 q_1^{-\alpha_1^{(2)}} q_2^{-\alpha_2^{(2)}}, \ldots)$$

G is dimension free since f is.

Thus

$$(3) \qquad G(q_1, q_2, \Pi_1, \Pi_2) = 0$$

is equivalent to (1). But (3) is unit free hence

$$G(\bar{q}_1, \bar{q}_2, \Pi_1, \Pi_2) = 0$$

for all $\lambda_i > 0$, $\bar{L}_i = \lambda_i L_i$,

$$(4) \qquad \bar{q}_1 = \lambda_1^{a_{11}} \lambda_2^{a_{21}} q_1$$

$$(5) \qquad \bar{q}_2 = \lambda_1^{a_{21}} \lambda_2^{a_{22}} q_2$$

In text it is shown that λ_1 and λ_2 can be chosen so that

$$\bar{q}_1 = 1 \qquad \bar{q}_2 = 1$$

Then (3) is equivalent to

$$F(\Pi_1, \Pi_2) \equiv G(1, 1, \Pi_1, \Pi_2) = 0 \qquad //$$

<u>Remark</u> That such a choice λ_1, λ_2 can be made is not entirely trivial and depends on the invertibilty of the submatrix

$$A_0 = \begin{bmatrix} a_{11} & a_{21} \\ a_{21} & a_{22} \end{bmatrix}$$