Inverse of Differential Operators

we recall the operator formulation for (SLP),

$$Ly \equiv -(p(x)y')' + q(x)y$$

with a domain defined using the boundary conditions

$$D = \{ y \in C^2[a,b] : B_1(y)=0,\ B_2(y)=0 \}$$

The eigenvalues of L are those λ such that

$$(1) \qquad Ly = \lambda y \qquad y \in D$$

has nontrivial solutions. If λ is NOT an eigenvalue then the only soln of (1) is the trivial solution $y(x) \equiv 0$.

Now consider the problem

$$(2) \qquad (L - \lambda I)y = f(x) \qquad y \in D$$

So long as λ is not an eigenvalue of the Sturm Liouville operator L, equation (2) has a unique solution and we may write

$$(3) \qquad y = (L - \lambda I)^{-1} f$$

where $(L-\lambda I)^{-1}$ is the operator that solves the problem.

So if $\lambda = 0$ is not an eigenvalue of L the unique solution $y(x)$ of

$$(4) \qquad L y = f(x) \qquad y \in D$$

may be written in operator form

$$(5) \qquad y = L^{-1} f \qquad y \in D$$

<u>Nonunique solutions</u>

If $\lambda = 0$ is an eigenvalue of L then the solution of (4) is not unique and it makes no sense to talk about an inverse operator. To see this suppose y_h is any nontrivial solution of

$$L y_h = 0 \qquad y_h \in D$$

and y_p be any "particular" solution of

$$L y_p = f \qquad y_p \in D$$

The $y = y_h + y_p \in D$ (satisfies boundary conds) and

$$L(y_h + y_p) = \cancelto{0}{L(y_h)} + L(y_p) = f$$

so y is another soln of (4).

Main point here is if $\lambda = 0$ an eigenvalue of L, problem (4) has many solutions so it makes no sense to discuss an inverse operator.

EXAMPLE A simple example demonstrating how one can explicitly define L^{-1}

(1) $$Lu \equiv u'' = f(x)$$

(2) $$u(0) = 0 \qquad u(1) = 0 \qquad \text{(Dirichet)}$$

Integrating (1) twice in x (constants c_1, c_2)

(3) $$u(x) = \int_0^x \int_0^t f(s)\,ds\,dt + c_1 x + \cancel{c_2}$$

Evaluating (3) at $x = 0$, $u(0) = c_2 = 0$.
Then the boundary condition $u(1) = 0$ implies

(4) $$c_1 = -\int_0^1 \int_0^t f(s)\,ds\,dt$$

Thus

$$u(x) = L^{-1} f = \int_0^x \int_0^t f(s)\,ds\,dt - x \int_0^1 \int_0^t f(s)\,ds\,dt$$

This one (of many) ways to explicitly define L^{-1}.

In this instance it is easy to see L^{-1} is a linear operator in the sense

$$L^{-1}(f_1 + f_2) = L^{-1} f_1 + L^{-1} f_2$$

Greens Functions for SLPs

Recall again the SLP

(1) $\quad Lu \equiv -(pu')' + qu = f(x)$

(2) $\quad B_1(u) = \alpha_1 u(a) + \beta_1 u'(a) = 0$

(3) $\quad B_2(u) = \alpha_2 u(b) + \beta_2 u'(b) = 0$

can be written compactly as

$$Lu = f \qquad u \in D$$

We seek a function $g(x,\xi)$ called a Green's function such that the soln of the SLP is

$$u(x) = L^{-1}f \equiv \int_a^b g(x,\xi) f(\xi)\, d\xi$$

Clearly $g(x,\xi)$ will depend on BOTH the form of L and the boundary conditions.

No such Greens function exists if $\lambda = 0$ is an eigenfalue of L.

Greens functions (once found) can be used to solve the (SLP) for ANY $f(x)$. Also demonstrates that the inverse of L is an integral operator with kernel $g(x,\xi)$. Also, L^{-1} is clearly linear.

Leibniz's Rule (General)

If $f(x,y)$ and $f_x(x,y)$ are (uniformly) continuous

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,\xi)\,d\xi = f(b,x)\frac{db}{dx} - f(a,x)\frac{da}{dx} + \int_{a(x)}^{b(x)} f_x(x,\xi)\,d\xi$$

A slightly less general form is (a constant)

$$\frac{d}{dx}\int_{a}^{x} f(x,\xi)\,d\xi = f(x,x) + \int_{a}^{x} f_x(x,\xi)\,d\xi$$

Proof outline of latter version

$$H(x) \equiv \int_a^x f(x,\xi)\,d\xi$$

$$H(x+\Delta x) = \int_a^{x+\Delta x} f(x+\Delta x,\xi)\,d\xi$$

Then

$$\frac{H(x+\Delta x) - H(x)}{\Delta x} = \frac{1}{\Delta x}\int_x^{x+\Delta x} f(x+\Delta x,\xi)\,d\xi + \int_a^x \frac{f(x+\Delta x,\xi) - f(x,\xi)}{\Delta x}\,d\xi$$

Let $\Delta x \to 0$ to get result. (First term MVT) /

Theorem Green's Functions for SLP

Suppose $\lambda = 0$ is not an eigenvalue of

(1) $Lu = -(pu')' + qu \qquad u \in D$

Then the solution of

(2) $Lu = f(x) \qquad u \in D$

is given by

(3) $u(x) = L^{-1} f \equiv \int_a^b g(x,\xi) f(\xi) d\xi$

where the Greens function $g(x,\xi)$ is

(4) $$g(x,\xi) = \begin{cases} g_+(x,\xi) = \dfrac{-u_1(x)\,u_2(\xi)}{p(\xi)\,W(\xi)} & x < \xi < b \\[2ex] g_-(x,\xi) = \dfrac{-u_1(\xi)\,u_2(x)}{p(\xi)\,W(\xi)} & a < \xi < x \end{cases}$$

where $u_k(x)$ are any independent solutions of

(5) $Lu_1 = 0 \qquad B_1(u_1) = 0 \qquad x = a$

(6) $Lu_2 = 0 \qquad B_2(u_2) = 0 \qquad x = b$

and $W(x)$ is the Wronskian

(7) $$W(x) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = u_1(x)\,u_2'(x) - u_1'(x)\,u_2(x)$$

EXAMPLE   Sample problem before proof

(1) $\quad Lu \equiv -u'' = f(x) \qquad x \in (0,1)$

(2) $\quad B_1(u) = u(0) = 0 \qquad$ Dirichlet

(3) $\quad B_2(u) = u(1) = 0 \qquad$ Dirichlet

First we will determine the Green's function using the theorem

For $Lu = -u''$ we have $p(x) \equiv 1$ since L is defined so that $Lu = -(pu')' + qu$.

Problems for $u_1(x)$, $u_2(x)$

$$-u_1'' = 0 \qquad u_1(0) = 0$$
$$-u_2'' = 0 \qquad u_2(1) = 0$$

have solutions $u_1(x) = x$, $u_2(x) = x - 1$

$$W(x) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \begin{vmatrix} x & x-1 \\ 1 & 1 \end{vmatrix} = 1$$

With $p(x) \equiv 1$, $W(x) \equiv 1$ the theorem yields

(4) $$g(x,\xi) = \begin{cases} g_+(x,\xi) = x(1-\xi) & x < \xi \\ g_-(x,\xi) = \xi(1-x) & \xi < x \end{cases}$$

Now we will demonstrate $u(x)$ defined by

(5) $$u(x) = \int_0^1 g(x,\xi) f(\xi)\, d\xi = L^{-1} f$$

solves the ODE.

To show $u(x)$ defined in (5) solves the SLP first rewrite it as follows

$$u(x) = \int_0^x g_-(x,\xi) f(\xi)\, d\xi + \int_x^1 g_+(x,\xi) f(\xi)\, d\xi$$

$$u(x) = u_-(x) + u_+(x)$$

Then Leibniz's rule $\Rightarrow$

$$u_-'(x) = x(1-\cancel{x}) f(x) - \int_0^x \xi f(\xi)\, d\xi$$

Similarly

$$u_+'(x) = -x(1-\cancel{x}) f(x) + \int_x^1 (1-\xi) f(\xi)\, d\xi$$

Adding we find

$$u'(x) = \int_x^1 f(\xi)\, d\xi - \underbrace{\int_0^1 \xi f(\xi)\, d\xi}_{\text{constant}}$$

Since the second term is constant

$$u''(x) = -f(x)$$

So $u(x)$ solves the diff eqn.

Moreover

$$g(0,\xi) = 0 \quad \Rightarrow \quad u(0) = 0$$
$$g(1,\xi) = 0 \quad \Rightarrow \quad u(1) = 0$$

So $u(x)$ also satisfies the SLP /

EXAMPLE Different operator and B. Cond.

(1) $Lu \equiv -(pu')' + qu = -u'' - u \qquad x \in (0, \pi)$

(2) $B_1(u) = u'(0) = 0$ Neumann B.C.

(3) $B_2(u) = u(\pi) = 0$ Dirichlet B.C.

Here $p(x) = 1$, $q(x) = -1$. Need solns of

$$u_1'' + u_1 = 0 \quad , \quad u_1'(0) = 0$$
$$u_2'' + u_2 = 0 \quad , \quad u_2(\pi) = 0$$

Choose

$$u_1(x) = \cos x \qquad u_2(x) = \sin x$$

Compute Wronskian (using $\sin^2 x + \cos^2 x = 1$)

$$W = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = +1$$

The Green's function is then

$$g(x, \xi) = \begin{cases} g_+(x, \xi) = -\cos x \sin \xi & x < \xi < b \\ g_-(x, \xi) = -\cos \xi \sin x & a < \xi < x \end{cases}$$

and the solution of

$$Lu = f \qquad u \in D$$

is

$$u(x) = \int_0^\pi g(x, \xi) f(\xi) \, d\xi$$

EXAMPLE   Failure

(1) $Lu = -u'' - u$ $x \in (0, \pi)$

(2) $B_1(u) = u(0) = 0$ Dirichlet

(3) $B_2(u) = u(\pi) = 0$ Dirichlet

One cannot use the Theorem to find a Green's function. The reason is
$\lambda = 0$ IS AN EIGENVALUE:

$$Lu = 0 \qquad u \in D$$

$$u'' + u = 0 \qquad u(0) = u(\pi) = 0$$

has nontrivial solutions $u(x) = A \sin x$.

Consequently

$$Lu = f \qquad u \in D$$

does not have a unique solution.

Had you tried to apply the Theorem

$$u_1'' + u_1 = 0 \qquad u_1(0) = 0 \qquad \Rightarrow \quad u_1(x) = \sin x$$
$$u_2'' + u_2 = 0 \qquad u_2(\pi) = 0 \qquad \Rightarrow \quad u_2(x) = \sin x$$

then the dependence of $u_K \Rightarrow W = 0$ and hence $g(x, \xi)$ is undefined.

Green's Function Properties

The Greens function is a function of both x and ξ. Regarding the differential operator L and Boundary operators $B_k(u)$ as operations in x we note. the following properties

(1) $L(g) = 0 \qquad x \neq \xi$

(2) $B_1(g) = 0$

(3) $B_2(g) = 0$

(4) $g(x,\xi)$ is continuous

(5) $g'(\xi^+,\xi) - g'(\xi^-,\xi) = -\dfrac{1}{p(\xi)}$

The first three are easy to see since for appropriately defined functions $a_k(\xi)$, $k=1,2$

$$g(x,\xi) = \begin{cases} a_2(\xi)\, u_1(x) & x<\xi \\ a_1(\xi)\, u_2(x) & x>\xi \end{cases}$$

So, if $x<\xi$

$$L(g) = L(a_2(\xi)u_1) = a_2(\xi)\,L(\cancel{u_1})^{0} = 0 \qquad x<\xi$$

$$B_1(g) = B_1(a_2 u_2) = a_2(\xi)\,B_1(\cancel{u_1})^{0} = 0 \qquad x<\xi$$

In a similar way the linearity of L, $B_2 \Rightarrow$

$$L(g) = a_1(\xi)\,L(\cancel{u_2})^{0} = 0 \qquad x>\xi$$

$$B_2(g) = a_1(\xi)\,\cancel{B_2(u_2)}^{0} = 0 \qquad x>\xi$$

Continuity of $g(x,\xi)$

$$A(\xi) = \frac{1}{p(\xi)W(\xi)}$$

$$g(x,\xi) = \begin{cases} -A(\xi)\,u_1(x)\,u_2(\xi) & x<\xi \\ -A(\xi)\,u_1(\xi)\,u_2(x) & x>\xi \end{cases}$$

then

$$g(\xi^+,\xi) = g(\xi^-,\xi) = -A(\xi)\,u_1(\xi)\,u_2(\xi)$$

Discont'y of $g'(x,\xi)$

$$(\;)' = \frac{d}{dx}(\;)$$

Explicitly

$$g'(x,\xi) = \begin{cases} g_+'(x,\xi) = \dfrac{-u_1'(x)u_2(\xi)}{p(\xi)W(\xi)} & x<\xi<b \\[2ex] g_-'(x,\xi) = \dfrac{-u_1(\xi)u_2'(x)}{p(\xi)W(\xi)} & a<\xi<x \end{cases}$$

From this it is evident

$$g'(\xi^+,\xi) - g'(\xi^-,\xi) = -\frac{(u_1(\xi)u_2'(\xi) - u_1'(\xi)u_2(\xi))}{p(\xi)\cancel{W(\xi)}}$$

$$= -\frac{1}{p(\xi)}$$

$x=\xi^+>\xi$
hence use
$g_-'(x,\xi)$
here

Proof of Theorem (at last)

Need to show that for $g(x,\xi)$ so defined

$$u(x) = \int_a^b g(x,\xi) f(\xi)\, d\xi$$

satisfies both boundary conditions and $Lu = f$. The former follows from the fact that $g(x,\xi)$ satisfies both B.C. in x, i.e.

$$B_k(u) = \int_a^b B_k(g) f(\xi)\, d\xi = 0 \qquad k = 1, 2.$$

Showing $Lu = f$ takes more work.

$$(1) \quad u(x) = \int_a^x g_-(x,\xi) f(\xi)\, d\xi + \int_x^b g_+(x,\xi) f(\xi)\, d\xi$$

Hence

$$u'(x) = \cancel{g_-(x,x) f(x)} + \int_a^x g_-'(x,\xi) f(\xi)\, d\xi$$

$$\cancel{- g_+(x,x) f(x)} + \int_x^b g_+'(x,\xi) f(\xi)\, d\xi$$

simplifies to

$$(2) \quad u'(x) = \int_a^x g_-'(x,\xi) f(\xi)\, d\xi + \int_x^b g_+'(x,\xi) f(\xi)\, d\xi$$

Using Leibniz's rule again

$$u''(x) = g_-'(x,x) f(x) + \int_a^x g_-''(x,\xi) f(\xi)\, d\xi$$

$$- g_+'(x,x) f(x) + \int_x^b g_+''(x,\xi) f(\xi)\, d\xi$$

which because of the jump discont^y of $g'(x,\xi)$ simplifies to

key term

$$(3) \quad u''(x) = -\frac{1}{p(x)} f(x) + \int_a^x g_-''(x,\xi) f(\xi)\, d\xi + \int_x^b g_+''(x,\xi) f(\xi)\, d\xi$$

Collectively, using (1)-(3) in Lu we find for

$$Lu = -pu'' - p'u' + qu$$

we get (formally)

$$Lu = f(x) + \int_a^x \cancelto{0}{L(g_-)} f(\xi)\, d\xi + \int_x^b \cancelto{0}{L(g_+)} f(\xi)\, d\xi$$

so that $Lu = f(x)$ $\square$

Eigenfunction Representation for $g(x,\xi)$

Let $\{\phi_n\}$ be the orthonormal efns satisfying

$$L\phi_n = \lambda_n \phi_n \qquad \phi_n \in D$$

and consider

$$(1) \qquad Lu = f \qquad u \in D$$

Expand out

$$(2) \qquad u(x) = \sum_{n=1}^{\infty} u_n \phi_n(x) \qquad u_n = \langle u, \phi_n \rangle$$

$$f(x) = \sum_{n=1}^{\infty} f_n \phi_n(x) \qquad f_n = \langle f, \phi_n \rangle$$

Since ϕ_n satisfy the boundary conditions so does $u(x)$ in (2). Then (1) $\Rightarrow$

$$\begin{aligned} Lu &= L\left(\sum_{n=1}^{\infty} u_n \phi_n\right) \\ &= \sum_{n=1}^{\infty} u_n L(\phi_n) \\ &= \sum_{n=1}^{\infty} u_n \lambda_n \phi_n(x) \\ &= \sum_{n=1}^{\infty} f_n \phi_n(x) \end{aligned}$$

Thus, orthogonality of $\phi_n(x)$ imply

$$u_n \lambda_n = f_n$$

Given $f(x)$ the series soln for $u(x)$ is

$$u(x) = \sum_{n=1}^{\infty} \frac{f_n}{\lambda_n} \phi_n(x) \quad , \quad \| \phi_n \| = 1$$

But $f_n = \langle f, \phi_n \rangle$ so

$$u(x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \int_a^b f(\xi) \phi_n(\xi) d\xi \cdot \phi_n(x)$$

Interchanging sum and integration order

$$u(x) = \int_a^b \underbrace{\left(\sum_{n=1}^{\infty} \frac{\phi_n(\xi) \phi_n(x)}{\lambda_n} \right)}_{g(x,\xi)} f(\xi) d\xi$$

So another way to represent $g(x,\xi)$ and L^{-1} is

$$u(x) = L^{-1} f = \int_a^b g(x,\xi) f(\xi) d\xi$$

where $g(x,\xi)$ given by the bilinear expansion

$$g(x,\xi) = \sum_{n=1}^{\infty} \frac{\phi_n(\xi) \phi_n(x)}{\lambda_n} \qquad \| \phi_n \| = 1$$

EXAMPLE Eigenfunction Representation

(1) $Lu = -u''$ $\quad x \in (0, \pi)$

(2) $u(0) = 0 \quad , \quad u(\pi) = 0$

Previously the normalized eigenfns were found as

$$\phi_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx) \qquad \lambda_n = n^2$$

Thus the solution of

$$Lu = f(x) \qquad u \in D$$

is given by

$$u(x) = \int_0^{\pi} g(x, \xi) f(\xi)\, d\xi$$

where the Green's function $g(x, \xi)$ is

$$g(x, \xi) = \sum_{n=1}^{\infty} \frac{\sin(nx) \sin(n\xi)}{n^2}$$

Note: series converges.

Connection to dirac delta function.

If $g(x,\xi)$ is the Green's function for L

$$(1) \qquad u(x) = \int_a^b g(x,\xi) f(\xi)\, d\xi$$

is the solution of

$$(2) \qquad Lu = f(x) \qquad\qquad u(x) \in D$$

Substituting (1) into (2)

$$Lu = L\left(\int_a^b g(x,\xi) f(\xi) d\xi\right) = \int_a^b (Lg) f(\xi)\, d\xi = f(x)$$

Technically the arrowed step is invalid since g is not differentiable at $x=\xi$. Lg is perfectly well defined for $x<\xi$ and $x>\xi$ just not at $x=\xi$. That being said we rewrite the last equality:

$$(3) \qquad \int_a^b (Lg)\, f(\xi)\, d\xi = f(x)$$

zero if $x \neq \xi$
yet acts
like a δ-function

Loosely speaking the dirac delta "function" $\delta(x)$ is defined by the properties

$$(4) \qquad \int_a^b \delta(x)\,dx = 1$$

$$(5) \qquad \int_a^b \delta(x) f(x)\,dx = f(0)$$

$$(6) \qquad \delta(x) = 0 \qquad \forall x \neq 0$$

Comparing these properties to (3)

$$\boxed{Lg = \delta(z-x) \qquad g \in D}$$

In many texts the Green's function is actually defined as the solution to this problem.

This motivates a more formal treatment of $\delta(x)$ and other "distributions".