Fredholm Integral Equations

Seek solns to Fredholm Integral Eqn (FIE)

$$\int_a^b k(x,y)\, u(y)\, dy \; - \lambda u(x) \; = f(x)$$

The function $k(x,y)$ is called the kernel and is assumed continuous, $k \in C[a,b]^2$.

If $f = 0$ the eqn is homogenous. Otherwise it is nonhomogeneous.

If $\lambda = 0$ the FIE is of the "First Kind". Otherwise it is of the "Second Kind".

Fredholm operator

$$\underline{K} u \equiv \int_a^b k(x,y)\, u(y)\, dy$$

For $k \in C[a,b]^2$ and $u \in L^2[a,b]$

$$\underline{K} : L^2 \rightarrow L^2$$

and (1) can be written in operator notation

$$(\underline{K} - \lambda \underline{I})\, u = f$$

where $\underline{I}$ is the identity operator.

If

$$(\underline{K} - \lambda I)\, u = f$$

has a unique solution one may write

$$u = (\underline{K} - \lambda I)^{-1} f$$

Solutions to such integral equations need not have unique solns

<u>Defn</u> If there is a $\phi_n(x) \in L^2[a,b]$ such that

$$\underline{K}\,\phi_n = \lambda_n \phi_n \qquad \phi_n \neq 0$$

for some λ_n then λ_n is an eigenvalue of $\underline{K}$ with associated eigenfn $\phi_n(x)$.

<u>Nonunique solns when $\lambda = \lambda_n$</u>

Suppose $\lambda = \lambda_n$ is an eigenvalue of $\underline{K}$ with eigenfunction $\phi_n(x)$. If u is any solution of $(\underline{K} - \lambda I)u = f$ then

$$\begin{aligned}(\underline{K} - \lambda_n I)(u + \alpha\phi_n) &= (\underline{K} - \lambda_n I)u + \alpha\,\cancelto{0}{(\underline{K} - \lambda_n I)\phi_n} \\ &= (\underline{K} - \lambda_n I)u \\ &= f\end{aligned}$$

shows that $u + \alpha\phi_n$ is also a soln $\forall \alpha \in \mathbb{R}$.

Degenerate Kernels

A kernel $k(x,y)$ is said to be degenerate if there are functions α_j and β_j s.t.

$$(1) \qquad k(x,y) = \sum_{j=1}^{n} \alpha_j(x)\beta_j(y) \quad , \qquad n < \infty$$

Such kernels are sometimes called separable.

EXAMPLE

$$f(x) + \lambda u(x) = \int_0^1 \underbrace{(x + 3x^2 y)}_{\text{sum of separable fns.}} u(y)\, dy$$

Solving Fredholm Eqns with Degenerate Kernels

If $k(x,y)$ is degenerate then

$$Ku - \lambda u = f(x)$$

expanded is

$$(2) \qquad \int_a^b \left(\sum_{j=1}^{n} \alpha_j(x)\beta_j(y) \right) u(y)\, dy = \lambda u(x) + f(x)$$

Reversing the integration/summation order and using inner product notation, (2) becomes

(3) $$\lambda u(x) = \sum_{j=1}^{n} \langle \beta_j, u \rangle \alpha_j(x) - f(x)$$

Defining

$$c_j \equiv \langle \beta_j, u \rangle$$

we then have

$$\boxed{(4) \quad \lambda u(x) = \sum_{j=1}^{n} c_j \alpha_j(x) - f(x)}$$

If we can determine the constants c_j then equation (4) yields the soln $u(x)$.

Multiply (4) by $\beta_i(x)$ and integrate over $[a,b]$

$$\lambda \underbrace{\langle \beta_i, u \rangle}_{c_i} = \sum_{j=1}^{n} c_j \langle \beta_i, \alpha_j \rangle - \langle \beta_i, f \rangle$$

Re-arrange. Then c_i are solutions of the linear system

(5) $$\boxed{\sum_{j=1}^{n} \langle \beta_i, \alpha_j \rangle c_j - \lambda c_i = \langle \beta_i, f \rangle}$$

This can be written as the matrix eqn

(6) $$(A - \lambda I)\vec{c} = \vec{F}$$

where $\vec{c} = (c_1, \ldots, c_n)^T$, $F_i = \langle \beta_i, f \rangle$ and $A \in \mathbb{R}^{n \times n}$ is

$$A = \begin{bmatrix} \langle \beta_1, \alpha_1 \rangle & \langle \beta_1, \alpha_2 \rangle & \cdots & \langle \beta_1, \alpha_n \rangle \\ \langle \beta_2, \alpha_1 \rangle & \cdot & \cdot & \cdot \\ \vdots & & & \\ \langle \beta_n, \alpha_1 \rangle & \cdot & \cdot & \langle \beta_n, \beta_n \rangle \end{bmatrix}$$

If one can solve (6) for $\vec{c}$ the soln of the integral eqn is given by (4)

<u>Remarks</u>

(a) If $(A - \lambda I)$ is invertible the soln is unique

(b) If λ is an eigenvalue of A $(A - \lambda I)$ is not invertible. In this case (6) either has no solution or nonunique solns.

If $\vec{F} \notin R(A - \lambda I)$ range/column space of $(A - \lambda I)$ then (6) has no soln.

EXAMPLE (Degenerate Kernel Soln)

Solve the integral equation

$$(1) \qquad 1 + \lambda u(x) = \int_0^1 (18x + 4x^2 y)\, u(y)\, dy$$

when $\lambda = 1$. Then find all the eigenvalues of the integral operator K.

$$f(x) + \lambda u(x) = \int_0^1 k(x,y)\, u(y)\, dy$$

The kernel $k(x,y)$ is separable

$$\alpha_1 = 18x \qquad \alpha_2 = 4x^2$$

$$\beta_1 = 1 \qquad \beta_2 = y$$

$$f = 1$$

Then

$$Ku = \int_0^1 \left\{ \sum_{j=1}^{2} \alpha_j(x) \beta_j(y) \right\} u(y)\, dy$$

The solution $u(x)$ of the integral eqn is (for $\lambda = 1$)

$$u(x) = \sum_{j=1}^{2} c_j \alpha_j(x) - 1$$

where $\vec{c} = (c_1, c_2)$ is a solution of

$$(2) \qquad (A - \lambda I)\,\vec{c} = \vec{F} \qquad \lambda = 1$$

and, after some calculations

$$A = \begin{bmatrix} \langle \beta_1, \alpha_1 \rangle & \langle \beta_1, \alpha_2 \rangle \\ \langle \beta_2, \alpha_1 \rangle & \langle \beta_2, \alpha_2 \rangle \end{bmatrix} = \begin{bmatrix} 9 & 4/3 \\ 6 & 1 \end{bmatrix}$$

$$\vec{F} = \begin{pmatrix} \langle \beta_1, f \rangle \\ \langle \beta_2, f \rangle \end{pmatrix} = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}$$

Written out explicitly with $\lambda = 1$, eqn (2) becomes

$$\begin{bmatrix} 8 & 4/3 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

whose soln is $c_1 = \frac{1}{12}$, $c_2 = \frac{1}{4}$.

Thus the soln $u(x)$ of the integral eqn (1) is

$$u(x) = c_1 \alpha_1(x) + c_2 \alpha_2(x) - 1$$

$$u(x) = \frac{3}{2} x + x^2 - 1$$

This solution is unique.

Finding the eigenvalues of $\underline{K}u$ amounts to finding the λ for which $\underline{K}u = \lambda u$ has a nontrivial soln.

Any solution of $(f \equiv 0)$

$$\int_0^1 k(x,y)\,u(y)\,dy = \lambda u(x)$$

can be written

$$u(x) = \sum_{j=1}^{2} c_j \alpha_j(x)$$

where $\vec{c} \in N(A-\lambda I)$ (Nullspace of $A-\lambda I$)

$$(A-\lambda I)\vec{c} = \vec{0} \qquad (f \equiv 0)$$

For our previously computed matrix A

$$\det(A-\lambda I) = P(\lambda) = \lambda^2 - 10\lambda + 1$$

is the characteristic polynomial of A.
Roots of $P(\lambda)$ are eigenvalues of $\underline{K}$!!

$$\lambda = \lambda_{\pm} = 5 \pm 2\sqrt{6}$$

For $\lambda = \lambda_+ = 5+2\sqrt{6}$, $\vec{c}_+ = (1, 6(\lambda_+-1)^{-1}) \in N(A-\lambda_+ I)$
so that

$$u_+(x) = \alpha_1(x) + 6(\lambda_+-1)^{-1}\alpha_2(x)$$

is an associated eigenfunction of $\underline{K}$

EXAMPLE Solve the integral equation

$$(1) \qquad \int_0^{\pi} (\sin x + \sin(2x)\cos y)\, u(y)\, dy = \lambda u(x) + f(x)$$

Again degenerate with $f(x) = \cos^2 x$

$$\alpha_1 = \sin x \qquad \alpha_2 = \sin 2x$$
$$\beta_1 = 1 \qquad \beta_2 = \cos y$$

Here the inner product

$$\langle u, v \rangle = \int_0^{\pi} u(x)\, v(x)\, dx$$

The solution $u(x)$ is given by

$$(2) \qquad \lambda u(x) = c_1 \alpha_1(x) + c_2 \alpha_2(x) - f(x)$$

where $\vec{c}$ is a solution of

$$(A - \lambda I)\vec{c} = \vec{F}$$

Here

$$(3) \qquad A = \begin{bmatrix} \langle \beta_1, \alpha_1 \rangle & \langle \beta_1, \alpha_2 \rangle \\ \langle \beta_2, \alpha_1 \rangle & \langle \beta_2, \alpha_2 \rangle \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 4/3 \end{bmatrix}$$

$$(4) \qquad \vec{F} = \begin{pmatrix} \langle \beta_1, f \rangle \\ \langle \beta_2, f \rangle \end{pmatrix} = \begin{pmatrix} \pi/2 \\ 0 \end{pmatrix}$$

Since A is diagonal its eigenvalues are just the diagonal elements

$$\lambda_1 = 2 \qquad \lambda_2 = 4/3$$

If $\lambda \neq \lambda_1, \lambda_2$ then $\vec{c}$ has a unique value. In fact $(A - \lambda I)\vec{c} = \vec{F}$ reduces to

$$(2-\lambda)\, c_1 = \frac{\pi}{2}$$

$$(4/3 - \lambda)\, c_2 = 0 \qquad (c_2 = 0)$$

So that

$$u(x) = c_1 \alpha_1(x) + \cancelto{0}{c_2} \alpha_2(x)$$

$$u(x) = \frac{\pi}{2(2-\lambda)} \sin x$$

Hilbert Schmidt Integral Operators

(1) $\quad \underline{K} u \equiv \int_a^b k(x,y)\, u(y)\, dy$

where

(2) $\quad k(x,y) \in C[a,b]^2 \qquad$ continuous, $L^2[a,b]^2$

(3) $\quad k(x,y) = k(y,x) \qquad$ symmetric kernel

(4) $\quad k(x,y)$ is not degenerate

Condition (3) defining Hilbert Schmidt operators implies the operator K is itself "symmetric"

(5) $\quad \langle \underline{K} u, v \rangle = \langle u, \underline{K} v \rangle \qquad \forall u, v \in L^2[a,b]$

Theorem Let $\underline{K}$ be Hilbert Schmidt. The $\underline{K}$ has infinitely many eigenvalues λ_K and associated eigenfunctions s.t.

(i) $\quad \lambda_K \in \mathbb{R}$

(ii) $\quad 0 < \cdots |\lambda_n| \leq \cdots \leq |\lambda_2| \leq |\lambda_1|$

(iii) $\quad \lim_{n \to \infty} |\lambda_n| = 0$

(iv)

$$f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x) \quad \text{for some } c_n, \text{ any } f \in L^2[a,b]$$

where $\{\phi_K\}$ are mutually orthogonal

The proof of this theorem is beyond the scope this course.

The symmetry property for the operator $\underline{K}$ is not hard to show:

$$\langle \underline{K}u, v\rangle = \int_a^b \left(\int_a^b k(x,y)u(y)dy\right) v(x)\,dx$$

$$= \int_a^b\int_a^b k(x,y)\,u(y)\,v(x)\,dy\,dx$$

Cont^y of $k(x,y)$ in particular

$$= \int_a^b u(y)\left(\int_a^b \underline{k(x,y)}\,v(x)\,dx\right)dy$$

$k(x,y)=k(y,x)$

$$= \int_a^b u(y)\left(\int_a^b \underline{k(y,x)}\,v(x)\,dx\right)dy$$

$$= \int_a^b u(y)\,(\underline{K}v)\,dy$$

$$= \langle u, \underline{K}v\rangle$$

<u>Remark</u> Though the Theorem parallels SL-theory for SL differential operators it is not practically useful since it is diffuct if not impossible to find eigenfunctions of non-degenerate integral operators.

Eigenfunction solutions to HS eqns

Let $\underline{K}$ be Hilbert Schmidt with (λ_n, ϕ_n) eval/efn pairs

$$\underline{K}\,\phi_n = \lambda_n \phi_n \qquad n = 1, 2, 3$$

The theorem guarantees $\{\phi_n\}$ are mutually orthogonal.

Seek a solution $u(x)$ of

$$(1) \qquad \int_a^b k(x,y)\,u(y)\,dy - \lambda\, u(x) = f(x)$$

In operator notation (1) is

$$(2) \qquad (\underline{K} - \lambda \underline{I})\, u = f$$

We expand using normalized efns $\hat{\phi}_n(x)$.

$$f(x) = \sum_{n=1}^{\infty} f_n \hat{\phi}_n(x) \qquad f_n = \langle f, \hat{\phi}_n \rangle$$

$$u(x) = \sum_{n=1}^{\infty} u_n \hat{\phi}_n(x) \qquad u_n = \langle f, \hat{\phi}_n \rangle$$

We shall show that (if λ not an eval of $\underline{K}$)

$$(3) \qquad u_k = \frac{f_k}{\lambda_k - \lambda} \qquad k = 1, 2, 3 \ldots$$

From the integral equation

$$\langle \hat{\phi}_m , \underline{K} u \rangle - \lambda \langle \hat{\phi}_m , u \rangle = \langle \hat{\phi}_m , f \rangle$$

$$\langle \hat{\phi}_m , \underline{K}\left(\sum_{n=1}^{\infty} u_n \hat{\phi}_n \right) \rangle - \lambda u_m = f_m$$

$\underline{K}\phi_n = \lambda_n \phi_n$

$$\langle \hat{\phi}_m , \sum_{n=1}^{\infty} u_n \lambda_n \hat{\phi}_n \rangle - \lambda u_m = f_m$$

now the orthogonality of $\hat{\phi}_n$ implies

$$u_m \lambda_m - \lambda u_m = f_m$$

Solving for u_m

$$u_m = \frac{f_m}{(\lambda_m - \lambda)}$$

Thus

$$(4) \qquad \boxed{u(x) = \sum_{m=1}^{\infty} \frac{f_m}{(\lambda_m - \lambda)} \hat{\phi}_m(x)}$$

where $\hat{\phi}_n(x)$ are the normalized efns.

<u>Remark</u> (a) Solution exists and unique if $\lambda \neq \lambda_m$

(b) Solution is $(\underline{K} - \lambda \underline{I})^{-1} f$.

(c) Dang hard to find $\hat{\phi}_m(x)$!

Dropping $\hat{}$ notation for normalized eigenfns $\phi_n(x)$ the solution $u(x)$ in (4) is

$$u(x) = \sum_{m=1}^{\infty} \frac{1}{(\lambda_m - \lambda)} \phi_m(x) \int_a^b f(y)\phi_m(y)\,dy$$

under sufficient convergence criteria we interchange integration/summation order

$$u(x) = \int_a^b \underbrace{\left(\sum_{m=1}^{\infty} \frac{\phi_m(x)\phi_m(y)}{\lambda_m - \lambda} \right)}_{l(x,y) \text{ kernel}} f(y)\,dy$$

Then

$$u(x) = (\underline{K} - \lambda \underline{I})^{-1} f$$

$$u(x) = L f$$

where the inverse integral operator $L = (\underline{K} - \lambda \underline{I})^{-1}$ is

$$L v \equiv \int_a^b l(x,y)\, v(y)\, dy$$

and also symmetric since $l(x,y) = l(y,x)$ above.

General Series Solutions (generally not possible)

First suppose λ is not an eigenvalue of $\underline{K}$ and let $\{\phi_n\}$ be any (complete) orthonormal set for $L^2[a,b]$.

Fredholm equation is

$$(1) \quad f(x) + \lambda u(x) = \int_a^b k(x,y)\, u(y)\, dy$$

Expand kernel as a double sum.

$$(2) \quad k(x,y) = \sum_{i=1}^{\infty} k_i(x)\, \phi_i(y)$$

where

$$(3) \quad k_i(x) = \int_a^b k(x,y)\, \phi_i(y)\, dy \quad , \quad \|\phi_i\| = 1$$

Each $k_i(x)$ has a series expansion as well

$$(4) \quad k_i(x) = \sum_{j=1}^{\infty} k_{ij}\, \phi_j(x)$$

where

$$(5) \quad k_{ij} = \int_a^b k_j(x)\, \phi_j(x)\, dx$$

are constants.

Using (3) in (5) we find

$$(6) \qquad k_{ij} = \int_a^b \int_a^b k(x,y)\,\phi_i(y)\,\phi_j(x)\,dy\,dx$$

Under suitable (uniform) convergence assumptions we have the following double sum representation for the kernel $k(x,y)$ using (4)-(6) in (2):

$$(7) \qquad k(x,y) = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} k_{ij}\,\phi_i(y)\,\phi_j(x)$$

We now seek a series solution

$$u(x) = \sum_{n=1}^{\infty} u_n \phi_n(x)$$

for the Fredholm eqn (1).

$$\underline{K} u = \int_a^b \left(\sum_{j=1}^{\infty} \sum_{i=1}^{\infty} k_{ij}\,\phi_i(y)\,\phi_j(x) \right) \left(\sum_{n=1}^{\infty} u_n \phi_n(y) \right) dy$$

using orthogonality of ϕ_i in $L^2[a,b]$

$$\underline{K} u = \sum_{j=1}^{\infty} \phi_j(x) \underbrace{\left(\sum_{n=1}^{\infty} k_{nj}\, u_n \right)}_{a_j}$$

Thus if

$$f(x) = \sum_{m=1}^{\infty} f_m \phi_m(x)$$

the integral equation (1) becomes

$$\sum_{j=1}^{\infty} (f_j + \lambda u_j)\phi_j(x) = \sum_{j=1}^{\infty} a_j \phi_j(x)$$

Orthogonality of ϕ_j then implies

$$f_j + \lambda u_j = \sum_{n=1}^{\infty} k_{nj} u_n$$

which <u>unfortunately</u> can't be solved for unknowns u_n.

$$\boxed{\text{Cannot solve general Fredholm equations using just any basis } \{\phi_n\} \text{ for } L^2[a,b].}$$