Multivariate Calculus Conventions and Theorems

Spatial Domains Ω

Throughout we shall let Ω denote a spatial domain and

Ω = open connected subset of $\mathbb{R}^n$ having a piecewise smooth boundary

We let $\bar{\Omega}$ be the closure of Ω so that

$$\bar{\Omega} = \Omega \cup \partial\Omega$$

EXAMPLE — Ω is the interior of a sphere. The sphere is its boundary.

EXAMPLE — $\Omega = (0,1)^3$ is the interior of the unit cube. $\bar{\Omega} = [0,1]^3$ is the closure. Although the boundary $\partial\Omega$ is not smooth it is piecewise smooth

EXAMPLE — If $\Omega = (a,b)$ then $\bar{\Omega} = [a,b]$ and the boundary $\partial\Omega$ is the set of endpoints

$$\partial\Omega = \{a, b\}$$

Divergence, Gradient, Curl, Laplacian

The del operator $\vec{\nabla}$ is the vector operator

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

and is used to define the following where $u = u(\vec{x})$ is a scalar function and $\vec{J} = (J_1, J_2, J_3)$ is a vector field

$$\vec{\nabla} u = (u_x, u_y, u_z) \qquad \underline{\text{gradient}}\ u$$

$$\vec{\nabla}\cdot\vec{J} = \frac{\partial J_1}{\partial x} + \frac{\partial J_2}{\partial y} + \frac{\partial J_3}{\partial z} \qquad \underline{\text{divergence}}\ \vec{J}$$

$$\vec{\nabla}\times\vec{J} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ J_1 & J_2 & J_3 \end{vmatrix} \qquad \text{curl}\ \vec{J}$$

$$\vec{\nabla}\cdot\vec{\nabla} u = \nabla^2 u = u_{xx} + u_{yy} + u_{zz} \qquad \underline{\text{Laplacian}}\ u$$

The Laplacian is sometimes written $\Delta u = \nabla^2 u$.

A couple of important identities are

$$\vec{\nabla}\times\vec{\nabla} u = 0$$

$$\vec{\nabla}\cdot(u\vec{J}) = \vec{\nabla} u\cdot\vec{J} + u\,\vec{\nabla}\cdot\vec{J} \qquad \underline{\text{product rule}}$$

The former states the curl of a conservative vector field $\vec{F} = \vec{\nabla} u$ vanishes.

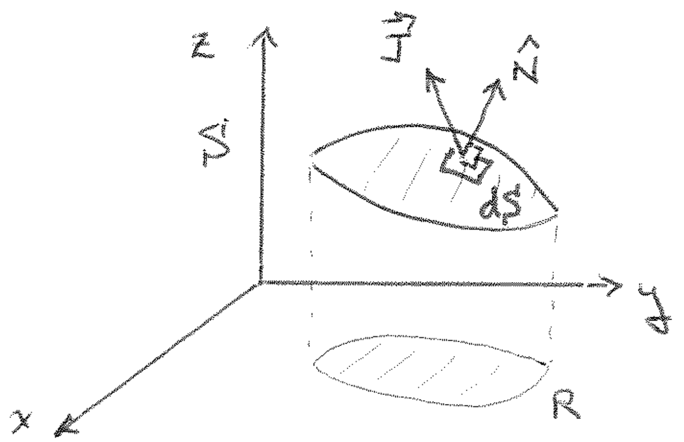
Flux

Let S be some (piecewise) smooth surface with unit normal $\hat{N}$ oriented in some unique way. Then the flux Φ of $\vec{J}(\vec{x}, t)$ thru S is

$$\Phi \equiv \iint_S \vec{J} \cdot \hat{N} \, dS$$

"net stuff per unit time thru S"

Here dS is a surface area element.



There are many ways to compute flux. If as above S is defined by a graph $z = f(x,y)$, $(x,y) \in R$ then

$$\Phi = \iint_R \vec{J} \cdot \vec{N} \Big|_{z = f(x,y)} dA$$

where $dA = dy\,dx$ is the area element in $\mathbb{R}^2$ and $\vec{N} = (-f_x, -f_y, 1)$ is the upward (nonunit) normal.

Divergence Theorem

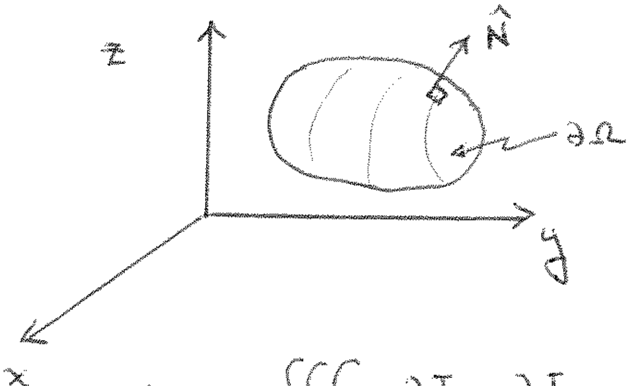
Let domain $\Omega \subset \mathbb{R}^n$ have boundary $\partial\Omega$ and $\hat{N}$ be the unit outward normal vector. Then

$$(1) \qquad \int_{\Omega} \vec{\nabla} \cdot \vec{J} \, d\vec{x} = \int_{\partial\Omega} \vec{J} \cdot \hat{N} \, dS$$

where $\vec{J} = (J_1, J_2, J_3)$ is sufficiently smooth.

Remark: We postpone issues regarding smoothness of $\vec{J}$ until later

Divergence Theorem in $\mathbb{R}^3$



unit normal $\hat{N}$

Here $J_K = J_K(x, y, z)$ generally.

$$(2) \qquad \iiint_{\Omega} \left(\frac{\partial J_1}{\partial x} + \frac{\partial J_2}{\partial y} + \frac{\partial J_3}{\partial z} \right) dV = \underbrace{\oiint_{\partial\Omega} \vec{J} \cdot \hat{N} \, dS}_{\text{net outward flux}}$$

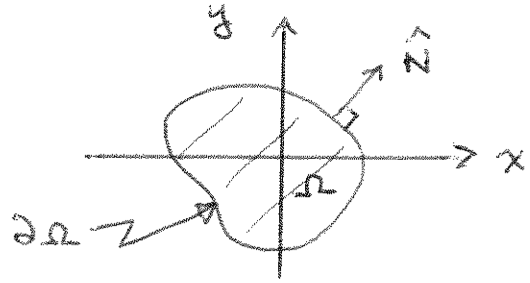
Here dV is a volume element and dS is a surface area element.

Divergence Theorem in $\mathbb{R}^2$

Here Ω is some domain in the xy-plane and $\partial\Omega$ is its bounding curve.

$$(3) \qquad \iint_{\Omega} \left(\frac{\partial J_1}{\partial x} + \frac{\partial J_2}{\partial y} \right) dA = \oint_{\partial\Omega} \vec{J} \cdot \hat{N} \, ds$$

Here dA is an area element in $\mathbb{R}^2$ and ds is the arclength element.



Given a parametrization of the boundary curve $(x(t), y(t)) = \underline{X}(t)$ the tangent and normal vectors are

$$\vec{T} = (\dot{x}, \dot{y}) \qquad \vec{N} = (-\dot{y}, \dot{x})$$

Using this and the fact the aclength element $ds = |\vec{T}| dt$ one can rewrite (3) as

$$(4) \qquad \iint_{\Omega} \left(\frac{\partial J_1}{\partial x} + \frac{\partial J_2}{\partial y} \right) dA = \oint_{\partial\Omega} J_2 \, dx - J_1 \, dy$$

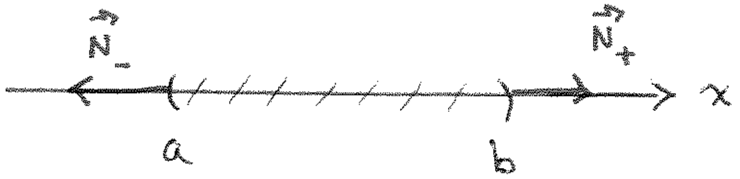
Contrast this with Green's Thm in the plane

$$(5) \qquad \iint_{\Omega} \left(\frac{\partial J_1}{\partial x} - \frac{\partial J_2}{\partial y} \right) dA = \oint_{\partial\Omega} J_2 \, dx + J_1 \, dy$$

They are the same if $(J_1, J_2) \mapsto (J_1, -J_2)$ and orientation on line integral reversed.

Divergence Theorem in $\mathbb{R}^1$

Here the domain Ω is some interval (a, b).



The outward normals are $\vec{N}_{\pm} = \pm \hat{i}$ and $\vec{J} = J_1(x)\hat{i}$. With these definitions

$$\int_{\Omega} \vec{\nabla} \cdot \vec{J} \, dx = \int_a^b \frac{dJ_1}{dx} \, dx$$

$$= J_1(b) - J_1(a)$$

$$= \vec{J} \cdot \hat{N}\Big|_{x=b} + \vec{J} \cdot \hat{N}\Big|_{x=a}$$

In other words, the Divergence Theorem in one dimension is the Fundamental Theorem of Calculus

$$\int_{\Omega} \vec{\nabla} \cdot \vec{J} \, dx = \underbrace{J_1(b) - J_1(a)}_{\text{Net flux out of } \Omega = (a,b)}$$

EXAMPLE Let Ω be the interior of the upper unit hemisphere. Define

$$\vec{J} = (J_1, J_2, J_3) = (xy^2, xz, zx^2)$$

We now use the Divergence Thm to compute the net flux Φ out of $\partial\Omega$.

$$\vec{\nabla} \cdot \vec{J} = \frac{\partial}{\partial x}(xy^2) + \cancel{\frac{\partial}{\partial y}(xz)} + \frac{\partial}{\partial z}(zx^2) = x^2 + y^2$$

Thus

$$\Phi = \iiint_{\Omega} (x^2 + y^2)\, dV$$

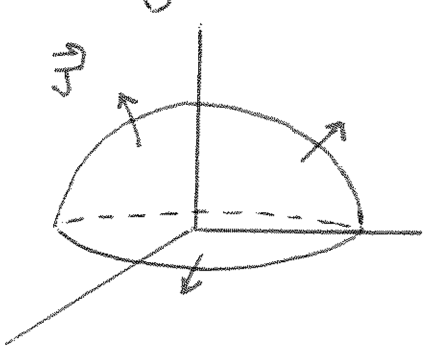
We use spherical coordinates to compute Φ

$$(1) \qquad \Phi = \int_0^{2\pi} \int_0^{\pi/2} \int_0^1 \underbrace{(\rho^2 \sin^2\phi)}_{x^2+y^2} \underbrace{\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta}_{dV}$$

where the limits define Ω

$$\Omega = \{(\rho, \phi, \theta) : \rho \in (0,1), \phi \in (0, \tfrac{\pi}{2}), \theta \in (0, 2\pi)\}$$

Evaluating (1) we find $\Phi = \frac{4\pi}{15}$.



Net flux out $\Phi = \frac{4\pi}{15}$

Divergence Theorem (General Version)

Let $u \in C^1(\bar{\Omega}) \cap C^2(\Omega)$. Then

$$(1) \qquad \int_{\Omega} \frac{\partial u}{\partial x_k}\, d\vec{x} = \int_{\partial\Omega} u \hat{N}_k\, ds$$

where $u = u(x_1, x_2, x_3)$ and $\hat{N} = (\hat{N}_1, \hat{N}_2, \hat{N}_3)$ is the outward unit normal

If you let $u = J_k$ in (1) and sum over $k = 1, 2, 3$ one obtains the usual form

$$\int_{\Omega} \vec{\nabla} \cdot \vec{J}\, d\vec{x} = \int_{\partial\Omega} \vec{J} \cdot \hat{N}\, ds$$

Lastly note the theorem is stated for a particular class of u.

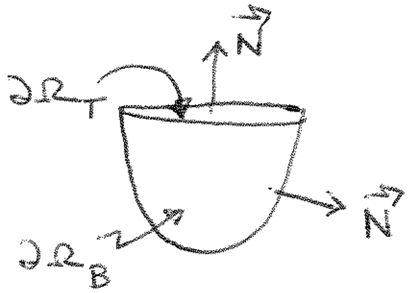
$C^1(\bar{\Omega})$ = set of u whose partial derivatives have continuous extensions to the closed set $\bar{\Omega}$ (contains boundary)

$C^2(\Omega)$ = set of u whose 2nd partials are all continuous of the open set Ω.

EXAMPLE Let Ω be bounded by $z=1$ and $z=x^2+y^2$. Choose $k=1$ and $u=x$ in Thm

$$(1) \qquad \int_{\Omega} \frac{\partial u}{\partial x}\, dV = \oint_{\partial\Omega} u \hat{N}_1 \, dS$$

Here



Normal Vectors

$\vec{N} = (0,0,1)$ on $\partial\Omega_T$

$\vec{N} = (2x, 2y, -1)$ on $\partial\Omega_B$

In particular $\hat{N}_1 = 0$ on $\partial\Omega_T$ so for $u=x$ the Divergence Thm (1) implies

$$\int_{\Omega} dV = \int_{\partial\Omega_B} 2x^2 \, dS$$

The nasty surface integral on the right is, then,

$$\int_{\partial\Omega_B} 2x^2 \, dS = \int_{\Omega} dV = \int_0^{2\pi}\int_0^1\int_{r^2}^1 r\,dz\,dr\,d\theta$$

$$= \frac{\pi}{2}$$