

Function Expansions - Examples

Let $f : [a, b] \rightarrow \mathbb{R}$. For various reasons one may want to represent $f(x)$ as a series

$$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x) \quad a_n \in \mathbb{R},$$

where $\phi_n(x)$ are "basis" functions

EXAMPLE Taylor Series

$$\phi_n(x) = (x - x_0)^n \quad a_n = \frac{f^{(n)}(x_0)}{n!}$$

If $f(x)$ and all its derivatives are continuous near x_0 , then Taylor's Thm implies

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$$

for all x near x_0 . In particular,

(a) The series converges

(b) the series converges to $f(x)$ for x near x_0

Some examples

$$e^x = 1 + x + \frac{1}{2!} x^2 + O(x^3) \quad x \in \mathbb{R}$$

$$\frac{1}{1-x} = 1 + x + x^2 + O(x^3) \quad |x| < 1$$

Note $f(x)$ may be defined for some x where the series is not, as in $(1-x)^{-1}$ above!

EXAMPLEFourier Series on $[-\pi, \pi]$ Theorem

Let $f(x)$ be piecewise smooth on $[-\pi, \pi]$ and define

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) dx \quad n \geq 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) dx \quad n \geq 1$$

Then at each $x \in (-\pi, \pi)$ at which $f(x)$ is continuous

$$(1) \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

This theorem is stated without proof and equality only applies at points of continuity.

For $f(x) = x$ it is easy to show $a_n = 0 \quad \forall n$ and

$$(2) \quad f(x) = \sum_{n=1}^{\infty} b_n \phi_n(x) = \sum_{n=1}^{\infty} \frac{(2(-1))^{n+1}}{n} \sin(nx)$$

where here $\phi_n(x) = \sin(nx)$ are "basis" fns.

For this example $f(x)$ note $f(\pi) = \pi$ does NOT equal the series (whose value is zero on account $\sin(n\pi) = 0$ for all n).

Function Expansion - Issues

$$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x)$$

A few points/issues

- (1) The \sim as opposed to $=$ symbol is sometimes to emphasize the fact that $f(x)$ does not equal the series at some x
- (2) What is a good choice of basis?
- (3) How does one determine the coefficients a_n once a choice of basis $\{\phi_n\}$ has been made?
- (4) How good an approximation is the truncated series?

$$f(x) \approx \sum_{n=0}^N a_n \phi_n(x)$$

how close are these?

The answers to some involve the study of complete orthogonal bases.

Square Integrable Functions $L^2[a, b]$

The theoretical framework for series representations of $f(x)$ is best formulated on the space

$$L^2[a, b] = \left\{ f : \int_a^b |f(x)|^2 dx < \infty \right\}$$

EXAMPLE Functions $f(x)$ continuous on the closed bounded interval $[a, b]$ are in $L^2[a, b]$

$$f(x) = \sin x \in L^2[0, \pi]$$

EXAMPLE $f(x) = x^p$, $p \neq \frac{1}{2}$ on $[0, 1]$

$$\begin{aligned} \int_0^1 |f(x)|^2 dx &= \frac{1}{2p+1} x^{2p+1} \Big|_0^1 \\ &= \frac{1}{2p+1} - \frac{1}{2p+1} \lim_{x \rightarrow 0^+} x^{2p+1} \end{aligned}$$

finite only if $p > -\frac{1}{2}$

Hence $x^{-\frac{1}{3}} \in L^2[0, 1]$ but $x^{-1} \notin L^2[0, 1]$

EXAMPLE Unbounded interval $[a, b] = \mathbb{R} = (-\infty, \infty)$

$$f(x) = e^{-|x|} \in L^2(\mathbb{R}) \text{ since } \int_{\mathbb{R}} e^{-2|x|} dx < \infty$$

$$f(x) = x^2 \notin L^2(\mathbb{R}) \text{ since } \int_{\mathbb{R}} x^2 dx \text{ unbounded}$$

Inner Products

Let \mathbb{X} be a linear space. An inner product on \mathbb{X} is a mapping that associates a real number $\langle x, y \rangle$ to each ordered pair $x, y \in \mathbb{X}$ and satisfies

$$(i) \quad \langle x, x \rangle \geq 0 \quad (\text{equality only if } x=0)$$

$$(ii) \quad \langle x, y \rangle = \langle y, x \rangle$$

$$(iii) \quad \langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$$

$$(iv) \quad \langle x, \alpha y \rangle = \alpha \langle x, y \rangle \quad \forall \alpha \in \mathbb{R}$$

EXAMPLE Dot product on $\mathbb{X} = \mathbb{R}^3$

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$$

for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$

EXAMPLE Inner product on $L^2[a, b]$

$$\langle f, g \rangle \equiv \int_a^b f(x) g(x) dx$$

is an inner product for $f(x) \in L^2[a, b]$, $g(x) \in L^2[a, b]$

As an example $f(x) = x$, $g(x) = x^2$ in $L^2[0, 1]$

$$\langle f, g \rangle = \int_0^1 x^3 dx = \frac{1}{4} x^4 \Big|_0^1 = \frac{1}{4}$$

Inner product, induced norm

Every inner product induces a norm

$$\|x\| = \sqrt{\langle x, x \rangle}$$

Recall that norms are used to determine distances and lengths:

$$\|x - y\| = \text{distance between } x, y$$

EXAMPLE Norm on \mathbb{R}^3 for $x = (x_1, x_2, x_3)$

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

EXAMPLE Norm on $L^2[a, b]$, $f(x) \in L^2[a, b]$

$$\|f\| = \sqrt{\langle f, f \rangle} = \left(\int_a^b |f(x)|^2 dx \right)^{1/2}$$

As an example if $f(x) = \cos x \in L^2[0, \pi]$

$$\langle f, f \rangle = \int_0^\pi \cos^2 x dx = \frac{\pi}{2} \Rightarrow \|f\| = \sqrt{\frac{\pi}{2}}$$

Another example, $f(x) = x, g(x) = x^2 \in L^2[0, 1]$

$$\|f - g\|^2 = \int_0^1 (x - x^2)^2 dx = \frac{1}{30}$$

so that

$$\|f - g\| = \sqrt{\frac{1}{30}} = \text{"distance between } f, g\text{"}$$

Orthogonal, Orthonormal

For any inner product space $x, y \in X$ are orthogonal if

$$\langle x, y \rangle = 0$$

If both x and y are unit length as well they are said to be orthonormal.

Every $x \in X$ can be normalized by dividing by its length

$$\hat{x} = \frac{x}{\|x\|} \Rightarrow \|\hat{x}\| = 1$$

EXAMPLE $x = (1, 2, 1)$ and $y = (-1, 1, -1)$ are orthogonal but not orthonormal in \mathbb{R}^3 , i.e.

$$\langle x, y \rangle = 0 \quad \langle x, x \rangle = 6 \neq 1$$

EXAMPLE $f(x) = x$ $g(x) = x(\alpha - x)$ $L^2[0, 1]$

$$\langle f, g \rangle = \int_0^1 x^2(\alpha - x) dx$$

$$\langle f, g \rangle = \frac{1}{3}\alpha - \frac{1}{4}$$

Thus f and g are \perp only if $\alpha = \frac{3}{4}$. Not orthonormal since

$$\|f\|^2 = \langle f, f \rangle = \int_0^1 x^2 dx = \frac{1}{3} \neq 1$$

EXAMPLE Orthonormal set on $L^2[0, \pi]$

$$f_n(x) = \sin(nx) \quad n = 1, 2, \dots$$

To compute $\langle f_n, f_m \rangle$ for integers n, m we use the identity

$$\sin A \sin B = \frac{1}{2} \cos(A-B) - \frac{1}{2} \cos(A+B)$$

For $n \neq m$ then

$$\langle f_n, f_m \rangle = \frac{1}{2} \int_0^\pi \cos((n-m)x) - \cos((n+m)x) dx$$

$$= \frac{1}{2(n-m)} \left[\sin((n-m)x) - \frac{1}{2(n+m)} \sin((n+m)x) \right]_0^\pi$$

$$= 0$$

For $n = m$

$$\langle f_n, f_n \rangle = \frac{1}{2} \int_0^\pi (1 - \cos 2nx) dx = \frac{\pi}{2}$$

so that $\|f_n\| = \sqrt{\frac{\pi}{2}}$. Summary

$$\langle f_n, f_m \rangle = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \end{cases}$$

So $\{f_n\}$ is an orthogonal set. $\{\hat{f}_n\}$ is an orthonormal set if

$$\hat{f}_n = \sqrt{\frac{2}{\pi}} \sin(nx)$$

Some Technical Details

That $L^2[a, b]$ is a linear space is not altogether obvious. In fact just because $f, g \in L^2[a, b]$ it is not obvious that the inner product $\langle f, g \rangle$ is defined!

Theorem Let $\langle x, y \rangle$ be an inner product on a linear space X . Then

$$(1) \quad |\langle x, y \rangle| \leq \|x\| \|y\| \quad (\text{Schwartz})$$

Schwartz inequality (1) assures that if $f, g \in L^2[a, b]$ the inner product $\langle f, g \rangle$ is bounded, i.e.

$$|\langle f, g \rangle| \leq \|f\| \cdot \|g\|$$

both finite if $f, g \in L^2$

Then

$$(f + g)^2 = f^2 + 2fg + g^2$$

can be integrated term by term

$$\|f + g\|^2 = \|f\|^2 + 2\langle f, g \rangle + \|g\|^2 < \infty$$

so that $f + g \in L^2[a, b]$.

Proof of Schwartz Expand out

$$\langle x - dy, x - dy \rangle \geq 0$$

$$d = \frac{\langle x, y \rangle}{\|y\|^2}$$

Parseval's Equality

If $\{f_n\}$ is an orthonormal (normalized orthogonal) sequence then we proved

$$(1) \quad e_N = \|f(x) - \sum_{n=1}^N c_n f_n(x)\|^2 = \|f\|^2 - \sum_{n=1}^N c_n^2$$

for any N if the Fourier coefficients

$$c_n = \langle f, f_n \rangle, \quad \|f_n\| = 1$$

If $e_N \rightarrow 0$ as $N \rightarrow \infty$ (more terms included)
then eqn (1) above implies

$$(2) \quad \|f\|^2 = \sum_{n=1}^N c_n^2 \quad \{f_n\} \text{ orthonormal}$$

which is known as Parseval's Equality.

In certain physical settings c_n^2 is the energy of the n -th "mode" and $\|f\|^2$ is the total energy.

Equation (2) can be used to derive formulae for certain infinite sums.

Pointwise versus L^2 (mean square) convergence

Let $\{f_n\}_{n \geq 1}$ be an orthogonal sequence on $L^2[a, b]$

$$\langle f_n, f_m \rangle = 0 \quad n \neq m$$

Now consider an N -term approximation $S_N(x)$ of $f(x)$ on $[a, b]$

$$S_N(x) \equiv \sum_{n=1}^N c_n f_n(x)$$

QUESTION How well does $S_N(x)$ approximate $f(x)$ as N gets large?

Toward answering this question we define two types of errors

$$E_N(x) \equiv f(x) - S_N(x)$$

$$e_N \equiv \|f - S_N\|^2 = \int_a^b |f(x) - S_N(x)|^2 dx$$

Here

$E_N(x)$ = pointwise error at $x \in [a, b]$

e_N = mean square error on $[a, b]$

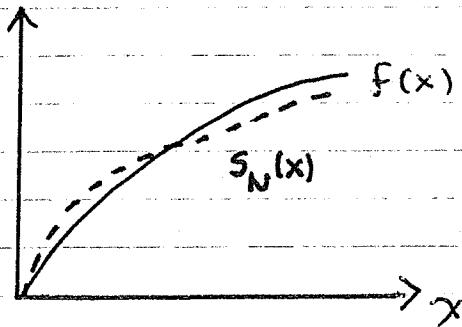
Defn we say $s_N \rightarrow f$ pointwise at $x \in [a, b]$ if

$$\lim_{N \rightarrow \infty} (f(x) - s_N(x)) = 0$$

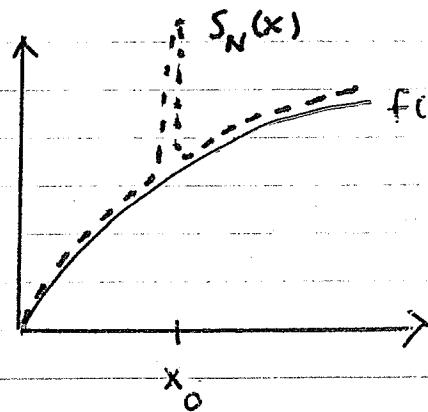
Defn we say $s_N \rightarrow f$ on $[a, b]$ in the L^2 or mean square sense if

$$\lim_{N \rightarrow \infty} \|f - s_N\|^2 = 0$$

The difference between these notions of convergence is subtle and best covered in an analysis class. If $s_N \rightarrow f$ in the L^2 sense it does in the pointwise sense "almost everywhere" which has a very precise meaning (measure theory). To see the difference



Here f and s_N are close in both the pointwise and L^2 sense



Here f and s_N are close in the L^2 sense but not pointwise at $x = x_0$

Generalized Fourier Series

Let $\{f_n(x)\}_{n \geq 1}$ be an orthogonal sequence.

If there are constants c_n such that

$$(1) \quad f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

in the L^2 sense then the infinite series is a generalized Fourier series representation of $f(x)$.

The constants c_n are called the Fourier coefficients.

Finding the Fourier Coefficients from (1)

Multiply (1) by $f_m(x)$ and integrate

$$\int_a^b f(x) f_m(x) dx = \sum_{n=1}^{\infty} c_n \int_a^b f_n(x) f_m(x) dx$$

0 unless $m=n$

$$\langle f, f_m \rangle = \sum_{n \geq 1} c_n \langle f_n, f_m \rangle$$

$$\langle f, f_m \rangle = c_m \|f_m\|^2$$

Hence

$$c_m = \frac{\langle f, f_m \rangle}{\|f_m\|^2} \quad \forall m \geq 1$$

Truncated Series Error e_N

Let $\{f_n\}$ be an orthogonal sequence and $S_N(x)$ be the N -term approximation of $f(x)$:

$$(1) \quad S_N(x) \equiv \sum_{n=1}^N c_n f_n(x)$$

A natural question to ask is what choice of Fourier coefficients c_n minimize the L^2 error:

$$e_N = e_N(c_1, \dots, c_N) = \|f(x) - S_N(x)\|^2$$

The error depends on c_1, \dots, c_N . We use calculus to minimize e_N

$$(2) \quad \frac{\partial e_N}{\partial c_K} = 0 \quad K = 1, 2, \dots, N$$

This is N eqns for N unknowns c_K .

$$\begin{aligned} \frac{\partial e_N}{\partial c_K} &= \frac{\partial}{\partial c_K} \int_a^b (f(x) - \sum_{n=1}^N c_n f_n(x))^2 dx \\ &= - \int_a^b 2(f(x) - \sum_{n=1}^N c_n f_n(x)) f_K(x) dx \\ &= -2 \langle f, f_K \rangle \stackrel{o \text{ if } n \neq k}{=} 2 \sum_{n=1}^N c_n \langle f_n, f_K \rangle \\ &= -2 \langle f, f_K \rangle \stackrel{o \text{ if } n \neq k}{=} 2 c_K \|f_K\|^2 \end{aligned}$$

Vanishes only if

$$(2) \quad c_K = \frac{\langle f, f_K \rangle}{\|f_K\|^2}$$

So now we know the choice

$$(3) \quad S_N(x) = \sum_{n=1}^N c_n f_n(x) \quad c_n = \frac{\langle f, f_n \rangle}{\|f_n\|^2}$$

minimizes the L^2 error $e_N = \|f - S_N\|^2$.

The next question is how big is this error?

$$e_N = \|f - S_N\|^2$$

$$= \langle f - S_N, f - S_N \rangle$$

$$= \langle f, f \rangle - 2 \langle f, S_N \rangle + \langle S_N, S_N \rangle$$

$$= \|f\|^2 - 2 \langle f, \sum_{n=1}^N c_n f_n \rangle + \left\langle \sum_{m=1}^N c_m f_m, \sum_{n=1}^N c_n f_n \right\rangle$$

$$= \|f\|^2 - 2 \sum_{n=1}^N c_n \underbrace{\langle f, f_n \rangle}_{\text{orthog}} + \sum_{n=1}^N c_n^2 \|f_n\|^2$$

$$= \|f\|^2 - 2 \sum_{n=1}^N c_n^2 \|f_n\|^2 + \sum_{n=1}^N c_n^2 \|f_n\|^2$$

So that

$$(4) \quad e_N = \|f\|^2 - \sum_{n=1}^N c_n^2 \|f_n\|^2$$

If additionally $\{f_n\}$ are normalized, i.e. $\|f_n\|=1$

$$(5) \quad e_N = \|f\|^2 - \sum_{n=1}^N c_n^2$$

EXAMPLE Previously we showed that

$$f_n(x) = \sin(nx) \quad \|f_n\| = \sqrt{\frac{2}{\pi}}$$

form an orthogonal set on $L^2[0, \pi]$

Let

$$\hat{f}_n = \sqrt{\frac{2}{\pi}} f_n \quad n \geq 1$$

so that $\{\hat{f}_n\}$ is an orthonormal set.

Assuming that

$$(1) \quad f(x) = x = \sum_{n=1}^{\infty} c_n \hat{f}_n(x)$$

for some c_n , derive a formula for the sum

$$I = \sum_{n=1}^{\infty} c_n^2$$

First compute the Fourier Coefficients

$$c_n = \langle f, \hat{f}_n \rangle = \sqrt{\frac{2}{\pi}} \int_0^\pi x \sin(nx) dx$$

$$c_n = \sqrt{\frac{2}{\pi}} \left(\frac{\sin(nx) - nx \cos(nx)}{n^2} \right) \Big|_0^\pi$$

$$c_n = \sqrt{\frac{2}{\pi}} \left(-\frac{\pi \cos(n\pi)}{n} \right)$$

$$(2) \quad c_n = \sqrt{2\pi} \frac{(-i)^{n+1}}{n}$$

These are relative to the normalized basis!

Parseval's Equality

$$\|f\|^2 = \sum_{n=1}^{\infty} c_n^2$$

Given $f(x) = x$ and the result (2) this becomes

$$\int_0^{\pi} x^2 dx = \sum_{n=1}^{\infty} \left(\sqrt{2\pi} \frac{(-1)^{n+1}}{n} \right)^2$$
$$I = \frac{1}{3} \pi^3 = 2\pi \sum_{n=1}^{\infty} \frac{1}{n^2}$$

From which

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$