Function Expansions - Examples

Let $f:[a,b]\to\mathbb{R}$. For various reasons one may want to represent $f(x)$ as a series

$$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x) \qquad a_n \in \mathbb{R}.$$

where $\phi_n(x)$ are "basis" functions

EXAMPLE Taylor Series

$$\phi_n(x) = (x-x_0)^n \qquad a_n = \frac{f^{(n)}(x_0)}{n!}$$

If $f(x)$ and all its derivatives are continuous near x_0 then Taylor's Thm implies

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

for all x near x_0. In particular,

(a) The series converges

(b) The series converges to $f(x)$ for x near x_0.

Some examples

$$e^x = 1 + x + \frac{1}{2!}x^2 + O(x^3) \qquad x\in\mathbb{R}$$

$$\frac{1}{1-x} = 1 + x + x^2 + O(x^3) \qquad |x|<1$$

Note $f(x)$ may be defined for some x where the series is not, as in $(1-x)^{-1}$ above!

EXAMPLE Fourier Series on $[-\pi, \pi]$

Theorem Let $f(x)$ be piecewise smooth on $[-\pi, \pi]$ and define

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \qquad n \geq 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \qquad n \geq 1$$

Then at each $x \in (-\pi, \pi)$ at which $f(x)$ is continuous

$$(1) \qquad f(x) = \frac{a_0}{2} + \sum_{n \geq 1}^{\infty} a_n\cos(nx) + b_n\sin(nx)$$

This theorem is stated without proof and equality <u>only applies at points of continuity</u>

For $f(x) = x$ it is easy to show $a_n = 0 \;\; \forall n$ and

$$(2) \qquad f(x) = \sum_{n=1}^{\infty} b_n \phi_n(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin(nx)$$

where here $\phi_n(x) = \sin(nx)$ are "basis" fns.

For this example $f(x)$ note $f(\pi) = \pi$ does NOT equal the series (whose value is zero on account $\sin(n\pi) = 0$ for all n).

Function Expansion - Issues

$$f(x) \sim \sum_{n=0}^{\infty} a_n \phi_n(x)$$

A few points / issues

(1) The $\sim$ as opposed to $=$ symbol is sometimes to emphasize the fact that $f(x)$ does not equal the series at some x

(2) what is a good choice of basis?

(3) How does one determine the coefficients a_n once a choice of basis $\{\phi_n\}$ has been made?

(4) How good an approximation is the truncated series?

$$\underbrace{f(x) \simeq \sum_{n=0}^{N} a_n \phi_n(x)}$$

how close are these?

The answers to some involve the study of complete orthogonal bases

Square Integrable Functions $L^2[a,b]$

The theoretical framework for series representations of $f(x)$ is best formulated on the space

$$L^2[a,b] = \left\{ f : \int_a^b f(x)^2 dx < \infty \right\}$$

EXAMPLE Functions $f(x)$ continuous on the closed bounded interval $[a,b]$ are in $L^2[a,b]$

$$f(x) = \sin x \in L^2[0,\pi]$$

EXAMPLE $f(x) = x^p$, $p \neq \frac{1}{2}$ on $[0,1]$

$$\int_0^1 f(x)^2 dx = \frac{1}{2p+1} x^{2p+1} \Big|_{0^+}^{1}$$

$$= \frac{1}{2p+1} - \frac{1}{2p+1} \underbrace{\lim_{x \to 0^+} x^{2p+1}}_{\text{finite only if } p > -\frac{1}{2}}$$

Hence $x^{-1/3} \in L^2[0,1]$ but $x^{-1} \notin L^2[0,1]$

EXAMPLE Unbounded interval $[a,b] = \mathbb{R} = (-\infty,\infty)$

$$f(x) = e^{-|x|} \in L^2(\mathbb{R}) \quad \text{since} \quad \int_{\mathbb{R}} e^{-2|x|} dx < \infty$$

$$f(x) = x^2 \notin L^2(\mathbb{R}) \quad \text{since} \quad \int_{\mathbb{R}} x^2 dx \ \text{unbounded}$$

Inner Products

Let $\mathbb{X}$ be a linear space. An inner product on $\mathbb{X}$ is a mapping that associates a real number $\langle x, y \rangle$ to each ordered pair $x, y \in \mathbb{X}$ and satisfies

(i) $\langle x, x \rangle \geq 0$ (equality only if $x = 0$)

(ii) $\langle x, y \rangle = \langle y, x \rangle$

(iii) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$

(iv) $\langle x, \alpha y \rangle = \alpha \langle x, y \rangle \quad \forall \alpha \in \mathbb{R}$

EXAMPLE Dot product on $\mathbb{X} = \mathbb{R}^3$

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3$$

for $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$

EXAMPLE Inner product on $L^2[a, b]$

$$\langle f, g \rangle \equiv \int_a^b f(x)\, g(x)\, dx$$

is an inner product for $f(x) \in L^2[a, b]$, $g(x) \in L^2[a, b]$

As an example $f(x) = x$, $g(x) = x^2$ in $L^2[0, 1]$

$$\langle f, g \rangle = \int_0^1 x^3 dx = \frac{1}{4} x^4 \Big|_0^1 = \frac{1}{4}$$

Inner product induced norm

Every inner product induces a norm

$$\| x \| = \sqrt{\langle x, x \rangle}$$

Recall that norms are used to determine distances and lengths:

$$\| x - y \| = \text{distance between } x, y$$

EXAMPLE Norm on $\mathbb{R}^3$ for $x = (x_1, x_2, x_3)$

$$\| x \| = \sqrt{\langle x, x \rangle} = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

EXAMPLE Norm on $L^2[a,b]$, $f(x) \in L^2[a,b]$

$$\| f \| = \sqrt{\langle f, f \rangle} = \left(\int_a^b f(x)^2 \, dx \right)^{1/2}$$

As an example if $f(x) = \cos x \in L^2[0, \pi]$

$$\langle f, f \rangle = \int_0^\pi \cos^2 x \, dx = \frac{\pi}{2} \qquad \Rightarrow \| f \| = \sqrt{\frac{\pi}{2}}$$

Another example, $f(x) = x$, $g(x) = x^2 \in L^2[0,1]$

$$\| f - g \|^2 = \int_0^1 (x - x^2)^2 \, dx = \frac{1}{30}$$

so that

$$\| f - g \| = \frac{1}{\sqrt{30}} = \text{"distance between } f, g\text{"}$$

Orthogonal, Orthonormal

For any inner product space $x, y \in \mathbb{X}$ are orthogonal if

$$\langle x, y \rangle = 0$$

If both x and y are unit length as well they are said to be orthonormal

Every $x \in \mathbb{X}$ can be normalized by dividing by its length

$$\hat{x} = \frac{x}{\|x\|} \Rightarrow \|\hat{x}\| = 1$$

EXAMPLE $x = (1, 2, 1)$ and $y = (-1, 1, -1)$ are orthogonal but not orthonormal in $\mathbb{R}^3$, i.e.

$$\langle x, y \rangle = 0 \qquad \langle x, x \rangle = 6 \neq 1$$

EXAMPLE $f(x) = x \qquad g(x) = x(\alpha - x) \qquad L^2[0, 1]$

$$\langle f, g \rangle = \int_0^1 x^2(\alpha - x)\, dx$$

$$\langle f, g \rangle = \frac{1}{3}\alpha - \frac{1}{4}$$

Thus f and g are $\perp$ only if $\alpha = \frac{3}{4}$.
Not orthonormal since

$$\|f\|^2 = \langle f, f \rangle = \int_0^1 x^2 dx = \frac{1}{3} \neq 1$$

EXAMPLE **Orthonormal set on $L^2[0,\pi]$**

$$f_n(x) = \sin(nx) \qquad n = 1, 2, \ldots$$

To compute $\langle f_n, f_m \rangle$ for integers n, m we use the identity

$$\sin A \sin B = \tfrac{1}{2}\cos(A-B) - \tfrac{1}{2}\cos(A+B)$$

For $n \neq m$ then

$$\langle f_n, f_m \rangle = \frac{1}{2}\int_0^\pi \cos((n-m)x) - \cos((n+m)x)\, dx$$

$$= \frac{1}{2(n-m)} \cancel{\sin((n-m)x)} - \frac{1}{2(n+m)} \cancel{\sin((n+m)x)} \Big|_0^\pi$$

$$= 0$$

For $n = m$

$$\langle f_n, f_n \rangle = \frac{1}{2}\int_0^\pi (1 - \cos 2nx)\, dx = \frac{\pi}{2}$$

so that $\| f_n \| = \sqrt{\frac{\pi}{2}}$. Summary

$$\langle f_n, f_m \rangle = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \end{cases}$$

So $\{f_n\}$ is an orthogonal set. $\{\hat{f}_n\}$ is an orthonormal set if

$$\hat{f}_n = \sqrt{\frac{2}{\pi}} \sin(nx)$$

Some Technical Details

That $L^2[a,b]$ is a linear space is not altogether obvious. In fact just because $f, g \in L^2[a,b]$ it is not obvious that the inner product $\langle f, g\rangle$ is defined!

Theorem Let $\langle x, y\rangle$ be an inner product on a linear space X. Then

$$(1) \qquad |\langle x,y\rangle| \le \|x\| \, \|y\| \qquad (\text{Schwartz})$$

Schwartz inequality (1) assures that if $f, g \in L^2[a,b]$ the inner product $\langle f, g\rangle$ is bounded, i.e.

$$|\langle f, g\rangle| \le \|f\| \cdot \|g\|$$

both finite if $f, g \in L^2$

Then

$$(f+g)^2 = f^2 + 2fg + g^2$$

can be integrated term by term

$$\|f+g\|^2 = \|f\|^2 + 2\langle f, g\rangle + \|g\|^2 < \infty$$

so that $f+g \in L^2[a,b]$.

Proof of Schwartz Expand out

$$\langle x - \alpha y, x - \alpha y\rangle \ge 0 \qquad \alpha \equiv \frac{\langle x, y\rangle}{\|y\|^2}$$

Pointwise versus L^2 (mean square) convergence

Let $\{f_n\}_{n \geq 1}$ be an orthogonal sequence on $L^2[a,b]$

$$\langle f_n, f_m \rangle = 0 \qquad n \neq m$$

Now consider an N-term approximation $S_N(x)$ of $f(x)$ on $[a,b]$

$$S_N(x) \equiv \sum_{n=1}^{N} c_n f(x)$$

QUESTION How well does $S_N(x)$ approximate $f(x)$ as N gets large?

Toward answering this question we define two types of errors

$$E_N(x) \equiv f(x) - S_N(x)$$

$$e_N \equiv \| f - S_N \|^2 = \int_a^b |f(x) - S_N(x)|^2 \, dx$$

Here

$E_N(x)$ = pointwise error at $x \in [a,b]$

$e_N(x)$ = mean square error on $[a,b]$

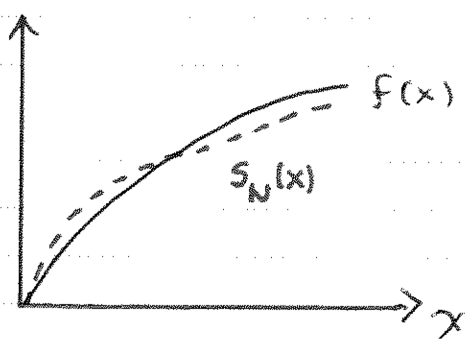
Defn We say $S_N \to f$ pointwise at $x \in [a,b]$ if

$$\lim_{N\to\infty} (f(x) - S_N(x)) = 0$$

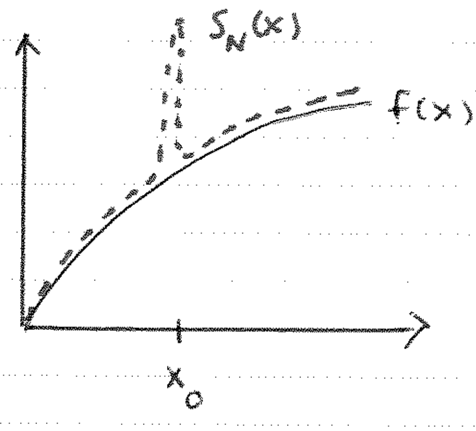
Defn We say $S_N \to f$ on $[a,b]$ in the L^2 or mean square sense if

$$\lim_{N\to\infty} \|f - S_N\|^2 = 0$$

The difference between these notions of convergence is subtle and best covered in an analysis class. If $S_N \to f$ in the L^2 sense it does in the pointwise sense "almost everywhere" which has a very precise meaning (measure theory). To see the difference



Here f and S_N are close in both the pointwise and L^2 sense



Here f and S_N are close in the L^2 sense but not pointwise at $x = x_0$

Generalized Fourier Series

Let $\{f_n(x)\}_{n\geq 1}$ be an orthogonal sequence.

If there are constants c_n such that

$$(1) \qquad f(x) = \sum_{n=1}^{\infty} c_n f_n(x)$$

in the L^2 sense then the infinite series is a generalized Fourier series representation of $f(x)$

The constants c_n are called the Fourier coefficients.

Finding the Fourier Coefficients from (1)

Multiply (1) by $f_m(x)$ and integrate

$$\int_a^b f(x) f_m(x)\,dx = \sum_{n=1}^{\infty} c_n \int_a^b f_n(x) f_m(x)\,dx$$

$$\langle f, f_m \rangle = \sum_{n\geq 1} c_n \underbrace{\langle f_n, f_m \rangle}_{0 \text{ unless } m=n}$$

$$\langle f, f_m \rangle = c_m \, \| f_m \|^2$$

Hence

$$\boxed{c_m = \frac{\langle f, f_m \rangle}{\| f_m \|^2} \qquad \forall m \geq 1}$$

Truncated Series Error e_N

Let $\{f_n\}$ be an orthogonal sequence and $S_N(x)$ be the N-term approximation of $f(x)$:

$$(1) \qquad S_N(x) \equiv \sum_{n=1}^{N} c_n f_n(x)$$

A natural question to ask is what choice of Fourier coefficients c_n minimize the L^2 error:

$$e_N = e_N(c_1, \ldots, c_N) = \| f(x) - S_N(x) \|^2$$

The error depends on $c_1 \ldots c_N$. We use calculus to minimize e_N

$$(2) \qquad \frac{\partial e_N}{\partial c_K} = 0 \qquad K = 1, 2 \ldots N$$

This is N eqns for N unknowns c_K.

$$\frac{\partial e_N}{\partial c_K} = \frac{\partial}{\partial c_K} \int_a^b \Big(f(x) - \sum_{n=1}^{N} c_n f_n(x) \Big)^2 dx$$

$$= \int_a^b 2\Big(f(x) - \sum_{n=1}^{N} c_n f_n(x) \Big) f_K(x)\, dx$$

$$= 2\langle f, f_K \rangle - 2 \sum_{n=1}^{N} c_n \overbrace{\langle f_n, f_K \rangle}^{0 \text{ if } n \neq k}$$

$$= 2\langle f, f_K \rangle - 2 c_K \| f_K \|^2$$

Vanishes only if

$$(2) \qquad \boxed{c_K = \frac{\langle f, f_K \rangle}{\| f_K \|^2}}$$

So now we know the choice

$$(3) \qquad S_N(x) = \sum_{n=1}^{N} c_n f_n(x) \qquad\qquad c_n = \frac{\langle f, f_n \rangle}{\| f_n \|^2}$$

minimizes the L^2 error $e_N = \| f - S_N \|^2$.

The next question is how big is this error?

$$\begin{aligned}
e_N &= \| f - S_N \|^2 \\
&= \langle f - S_N , f - S_N \rangle \\
&= \langle f, f \rangle - 2 \langle f, S_N \rangle + \langle S_N , S_N \rangle \\
&= \| f \|^2 - 2 \langle f, \sum_{n=1}^{N} c_n f_n \rangle + \underbrace{\langle \sum_{m=1}^{N} c_m f_m , \sum_{n=1}^{N} c_n f_n \rangle}_{\text{orthog.}} \\
&= \| f \|^2 - 2 \sum_{n=1}^{N} c_n \langle f, f_n \rangle + \sum_{n=1}^{N} c_n^2 \| f_n \|^2 \\
&= \| f \|^2 - 2 \sum_{n=1}^{N} c_n^2 \| f_n \|^2 + \sum_{n=1}^{N} c_n^2 \| f_n \|^2
\end{aligned}$$

So that

$$(4) \qquad e_N = \| f \|^2 - \sum_{n=1}^{N} c_n^2 \| f_n \|^2$$

If additionally $\{f_n\}$ are normalized, i.e $\| f_n \| = 1$

$$(5) \qquad e_N = \| f \|^2 - \sum_{n=1}^{N} c_n^2$$

Parseval's Equality

If $\{f_n\}$ is an orthonormal (normalized orthogonal) sequence then we proved

$$(1) \quad e_N = \| f(x) - \sum_{n=1}^{N} c_n f_n(x) \|^2 = \| f \|^2 - \sum_{n=1}^{N} c_n^2$$

for any N if the Fourier coefficients

$$c_n = \langle f, f_n \rangle \quad , \qquad \| f_n \| = 1$$

If $e_N \to 0$ as $N \to \infty$ (more terms included) then eqn (1) above implies

$$(2) \qquad \boxed{\| f \|^2 = \sum_{n=1}^{N} c_n^2} \qquad \{f_n\} \text{ orthonormal}$$

which is known as Parseval's Equality.

In certain physical settings c_n^2 is the energy of the n-th "mode" and $\| f \|^2$ is the total energy.

Equation (2) can be used to derive formulae for certain infinite sums.

EXAMPLE Previously we showed that

$$f_n(x) = \sin(nx) \qquad \|f_n\| = \sqrt{\frac{\pi}{2}}$$

form an orthogonal set on $L^2[0,\pi]$
Let

$$\hat{f}_n = \sqrt{\frac{2}{\pi}}\, f_n \qquad n \geq 1$$

so that $\{\hat{f}_n\}$ is an orthonormal set.

Assuming that

$$(1) \qquad f(x) = x = \sum_{n=1}^{\infty} c_n \hat{f}_n(x)$$

for some c_n, derive a formula for the sum

$$\underline{I} = \sum_{n=1}^{\infty} c_n^2$$

First compute the Fourier Coefficients

$$c_n = \langle f, \hat{f}_n \rangle = \sqrt{\frac{2}{\pi}} \int_0^{\pi} x \sin(nx)\, dx$$

$$c_n = \sqrt{\frac{2}{\pi}} \left(\frac{\cancelto{0}{\sin(nx)} - nx\cos(nx)}{n^2} \right) \Bigg|_0^{\pi}$$

$$c_n = \sqrt{\frac{2}{\pi}} \left(-\frac{\pi \cos(n\pi)}{n} \right)$$

$$(2) \qquad c_n = \sqrt{2\pi}\, \frac{(-1)^{n+1}}{n}$$

These are relative to the <u>normalized</u> basis!

Parseval's Equality

$$\|f\|^2 = \sum_{n=1}^{\infty} c_n^2$$

Given $f(x) = x$ and the result (2) this becomes

$$\int_0^{\pi} x^2 dx = \sum_{n=1}^{\infty} \left(\sqrt{2\pi} \frac{(-1)^{n+1}}{n} \right)^2$$

$$I = \frac{1}{3} \pi^3 = 2\pi \sum_{n=1}^{\infty} \frac{1}{n^2}$$

From which

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

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