Function Expansions - Examples

Let \( f: [a, b] \to \mathbb{R} \). For various reasons one may want to represent \( f(x) \) as a series

\[
f(x) \sim \sum_{n=0}^{\infty} a_n \varphi_n(x) \quad \text{where} \quad a_n \in \mathbb{R}.
\]

where \( \varphi_n(x) \) are "basis" functions.

EXAMPLE. Taylor Series

\[
\varphi_n(x) = (x - x_0)^n \quad a_n = \frac{f^{(n)}(x_0)}{n!}.
\]

If \( f(x) \) and all its derivatives are continuous near \( x_0 \), then Taylor's Thm implies

\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n
\]

for all \( x \) near \( x_0 \). In particular,

(a) The series converges

(b) The series converges to \( f(x) \) for \( x \) near \( x_0 \).

Some examples

\[
e^x = 1 + x + \frac{1}{2!} x^2 + O(x^3) \quad x \in \mathbb{R}
\]

\[
\frac{1}{1-x} = 1 + x + x^2 + O(x^3) \quad |x| < 1
\]

Note \( f(x) \) may be defined for some \( x \) where the series is not, as in \( (1-x)^{-1} \) above.
EXAMPLE  

Fourier Series on $[-\pi, \pi]$

Theorem  

Let $f(x)$ be piecewise smooth on $[-\pi, \pi]$ and define

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx) \, dx \quad n \geq 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx) \, dx \quad n \geq 1$$

Then at each $x \in (-\pi, \pi)$ at which $f(x)$ is continuous,

$$(1) \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

This theorem is stated without proof and equality only applies at points of continuity.

For $f(x) = x$ it is easy to show $a_n = 0 \ \forall n$ and

$$(2) \quad f(x) = \sum_{n=1}^{\infty} b_n \phi_n(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)$$

where here $\phi_n(x) = \sin(nx)$ are "basis" functions.

For this example $f(x)$ note $f(\pi) = \pi$ does not equal the series (whose value is zero on account $\sin(n\pi) = 0$ for all $n$).
Function Expansion - Issues

\[ f(x) \approx \sum_{n=0}^{\infty} a_n \phi_n(x) \]

A few points/issues

1. The \( \approx \) as opposed to \( = \) symbol is sometimes to emphasize the fact that \( f(x) \) does not equal the series at some \( x \).

2. What is a good choice of basis?

3. How does one determine the coefficients once a choice of basis \( \{ \phi_n \} \) has been made?

4. How good an approximation is the truncated series?

\[ f(x) \approx \sum_{n=0}^{N} a_n \phi_n(x) \]

how close are these?

The answers to some involve the study of complete orthogonal bases.
Square Integrable Functions $L^2[a,b]$

The theoretical framework for series representations of $f(x)$ is best formulated on the space

$$L^2[a,b] = \{f : \int_a^b |f(x)|^2 \, dx < \infty\}$$

**Example** Functions $f(x)$ continuous on the closed bounded interval $[a, b]$ are in $L^2[a,b]$

$$f(x) = \sin x \in L^2[0, \pi]$$

**Example** $f(x) = x^p$, $p \neq \frac{1}{2}$ on $[0, 1]$

$$\int_0^1 f(x)^2 \, dx = \int_0^1 x^{2p+1} \, dx = \left[ \frac{x^{2p+2}}{2p+2} \right]_0^1$$

$$= \frac{1}{2p+1} - \frac{1}{2p+1} \lim_{x \to 0^+} x^{2p+1}$$

finite only if $p > -\frac{1}{2}$

Hence $x^{-\frac{3}{2}} \in L^2[0,1]$ but $x^{-1} \notin L^2[0,1]$

**Example** Unbounded interval $[a,b] = \mathbb{R} = (-\infty, \infty)$

$$f(x) = e^{-x^2} \in L^2(\mathbb{R}) \text{ since } \int_{\mathbb{R}} e^{-2x^2} \, dx < \infty$$

$$f(x) = x^2 \notin L^2(\mathbb{R}) \text{ since } \int_{\mathbb{R}} x^2 \, dx \text{ unbounded}$$
Inner Products

Let \( \mathbf{X} \) be a linear space. An inner product on \( \mathbf{X} \) is a mapping that associates a real number \( \langle x, y \rangle \) to each ordered pair \( x, y \in \mathbf{X} \) and satisfies

(i) \( \langle x, x \rangle \geq 0 \) (equality only if \( x = 0 \))

(ii) \( \langle x, y \rangle = \langle y, x \rangle \)

(iii) \( \langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle \)

(iv) \( \langle x, \lambda y \rangle = \lambda \langle x, y \rangle \) \quad \forall x \in \mathbf{X}

EXAMPLE Dot product on \( \mathbf{X} = \mathbb{R}^3 \)

\[ \langle x, y \rangle = x_1 y_1 + x_2 y_2 + x_3 y_3 \]

for \( x = (x_1, x_2, x_3) \) and \( y = (y_1, y_2, y_3) \)

EXAMPLE Inner product on \( L^2[a, b] \)

\[ \langle f, g \rangle = \int_a^b f(x) g(x) \, dx \]

is an inner product for \( f(x) \in L^2[a, b], g(x) \in L^2[a, b] \)

As an example \( f(x) = x, \quad g(x) = x^2 \) in \( L^2[0, 1] \)

\[ \langle f, g \rangle = \int_0^1 x^3 \, dx = \frac{1}{4} x^4 \bigg|_0^1 = \frac{1}{4} \]
Inner product induced norm

Every inner product induces a norm

$$\|x\| = \sqrt{\langle x, x \rangle}$$

Recall that norms are used to determine distances and lengths:

$$\|x - y\| = \text{distance between } x, y$$

**Example** Norm on $\mathbb{R}^3$ for $x = (x_1, x_2, x_3)$

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

**Example** Norm on $L^2[a, b]$ , $f(x) \in L^2[a, b]$

$$\|f\| = \sqrt{\langle f, f \rangle} = \left( \int_a^b |f(x)|^2 \, dx \right)^{1/2}$$

As an example, if $f(x) = \cos x \in L^2[0, \pi]$\n
$$\langle f, f \rangle = \int_0^\pi \cos^2 x \, dx = \frac{\pi}{2}$$

$$\|f\| = \sqrt{\frac{\pi}{2}}$$

Another example , $f(x) = x, g(x) = x^2 \in L^2[0, 1]$

$$\|f - g\|^2 = \int_0^1 (x - x^2)^2 \, dx = \frac{1}{30}$$

so that

$$\|f - g\| = \frac{1}{\sqrt{30}} = \text{"distance between } f, g \text{"}$$
Orthogonal, Orthonormal

For any inner product space $x, y \in X$ are orthogonal if

$$\langle x, y \rangle = 0$$

If both $x$ and $y$ are unit length as well they are said to be orthonormal.

Every $x \in X$ can be normalized by dividing by its length

$$\hat{x} = \frac{x}{\|x\|} \Rightarrow \|\hat{x}\| = 1$$

**Example**

$x = (1, 2, 1)$ and $y = (-1, 1, -1)$ are orthogonal but not orthonormal in $\mathbb{R}^3$, i.e.

$$\langle x, y \rangle = 0 \quad \langle x, x \rangle = 6 \neq 1$$

**Example**

$f(x) = x$ and $g(x) = x(x-x)$ \(L^2[0,1]\)

$$\langle f, g \rangle = \int_0^1 x^2(x-x) \, dx$$

$$\langle f, g \rangle = \frac{1}{3} x - \frac{1}{4}$$

Thus $f$ and $g$ are only if $x = \frac{3}{4}$.

Not orthonormal since

$$\|f\|^2 = \langle f, f \rangle = \int_0^1 x^2 \, dx = \frac{1}{3} \neq 1$$
EXAMPLE Orthogonal set on $L^2[0, \pi]$

$$f_n(x) = \sin(nx), \quad n = 1, 2, \ldots$$

To compute $\langle f_n, f_m \rangle$ for integers $n, m$, we use the identity

$$\sin A \sin B = \frac{1}{2} \cos(A-B) - \frac{1}{2} \cos(A+B)$$

For $n \neq m$ then

$$\langle f_n, f_m \rangle = \frac{1}{2} \int_0^\pi \cos((n-m)x) - \cos((n+m)x) \, dx$$

$$= \frac{1}{2} \left[ \frac{\sin((n-m)x)}{d(n-m)} - \frac{\sin((n+m)x)}{d(n+m)} \right]_0^\pi$$

$$= 0$$

For $n = m$

$$\langle f_n, f_n \rangle = \frac{1}{2} \int_0^\pi (1 - \cos 2nx) \, dx = \frac{\pi}{2}$$

so that $\|f_n\| = \sqrt{\frac{\pi}{2}}$. Summary

$$\langle f_n, f_m \rangle = \begin{cases} 
0 & n \neq m \\
\frac{\pi}{2} & n = m
\end{cases}$$

so $\{f_n\}$ is an orthogonal set. If $\{f_n\}$ is an orthonormal set if

$$\hat{f}_n = \sqrt{\frac{2}{\pi}} \sin(nx)$$
Some Technical Details

That $L^2[a,b]$ is a linear space is not altogether obvious. In fact, just because $f, g \in L^2[a,b]$ it is not obvious that the inner product $\langle f, g \rangle$ is defined!

**Theorem** Let $\langle x, y \rangle$ be an inner product on a linear space $X$. Then

$$1 \langle x, y \rangle \leq ||x|| \cdot ||y|| \quad \text{(Schwartz)}$$

Schwartz inequality (1) assures that if $f, g \in L^2[a,b]$ the inner product $\langle f, g \rangle$ is bounded, i.e.

$$|\langle f, g \rangle| \leq ||f|| \cdot ||g||$$

Both finite if $f, g \in L^2$

Then

$$(f + g)^2 = f^2 + 2f \cdot g + g^2$$

can be integrated term by term

$$||f + g||^2 = ||f||^2 + 2\langle f, g \rangle + ||g||^2 < \infty$$

So that $f + g \in L^2[a,b]$.

**Proof of Schwarz** Expand out

$$\langle x - \alpha y, x - \alpha y \rangle \geq 0$$

$$\alpha = \frac{\langle x, y \rangle}{||y||^2}$$
Pointwise versus $L^2$ (mean square) convergence

Let $\{f_n\}_{n=1}^\infty$ be an orthogonal sequence on $L^2[a,b]$ such that

$$\langle f_n, f_m \rangle = 0 \quad n \neq m$$

Now consider an $N$-term approximation $S_N(x)$ of $f(x)$ on $[a, b]$:

$$S_N(x) \equiv \sum_{n=1}^{N} c_n f(x)$$

**Question:** How well does $S_N(x)$ approximate $f(x)$ as $N$ gets large?

Toward answering this question we define two types of errors:

$$E_N(x) \equiv f(x) - S_N(x)$$

$$e_N \equiv \|f - S_N\|_2^2 = \int_a^b |f(x) - S_N(x)|^2 \, dx$$

Here:

$E_N(x)$ = pointwise error at $x \in [a, b]$

$e_N(x)$ = mean square error on $[a, b]$
**Defn** We say \( S_N \to f \) pointwise at \( x \in [a, b] \) if

\[
\lim_{N \to \infty} (f(x) - S_N(x)) = 0
\]

**Defn** We say \( S_N \to f \) on \([a, b]\) in the \( L^2 \) or mean square sense if

\[
\lim_{N \to \infty} ||f - S_N||^2 = 0
\]

The difference between these notions of convergence is subtle and best covered in an analysis class. If \( S_N \to f \) in the \( L^2 \) sense it does in the pointwise sense "almost everywhere" which has a very precise meaning (measure theory).

To see the difference:

![Graph](https://via.placeholder.com/150)

Here \( f \) and \( S_N \) are close in both the pointwise and \( L^2 \) sense.

![Graph](https://via.placeholder.com/150)

Here \( f \) and \( S_N \) are close in the \( L^2 \) sense but not pointwise at \( x = x_0 \).
Generalized Fourier Series

Let \( \{f_n(x)\}_{n \geq 1} \) be an orthogonal sequence.

If there are constants \( c_n \) such that

\[
(1) \quad f(x) = \sum_{n=1}^{\infty} c_n f_n(x)
\]

in the \( L^2 \) sense then the infinite series is a generalized Fourier series representation of \( f(x) \).

The constants \( c_n \) are called the Fourier coefficients.

Finding the Fourier Coefficients from (1)

Multiply (1) by \( f_m(x) \) and integrate

\[
\int_a^b f(x)f_m(x)\,dx = \sum_{n=1}^{\infty} c_n \int_a^b f_n(x)f_m(x)\,dx
\]

\[
\langle f, f_m \rangle = \sum_{n \geq 1} c_n \langle f_n, f_m \rangle
\]

\[
\langle f, f_m \rangle = c_m \|f_m\|^2
\]

Hence

\[
c_m = \frac{\langle f, f_m \rangle}{\|f_m\|^2}, \quad \forall m \geq 1
\]
Truncated Series Error $e_N$

Let $\{f_n\}$ be an orthogonal sequence and $S_N(x)$ be the N-term approximation of $f(x)$:

$$S_N(x) = \sum_{n=1}^{N} c_n f_n(x)$$

A natural question to ask is what choice of Fourier coefficients $c_n$ minimize the $L^2$ error:

$$e_N = e_N(c_1, \ldots, c_N) = \|f(x) - S_N(x)\|^2$$

The error depends on $c_1, \ldots, c_N$. We use calculus to minimize $e_N$.

$$\frac{\partial e_N}{\partial c_k} = 0 \quad K = 1, 2, \ldots, N$$

This is $N$ eqns for $N$ unknowns $c_k$.

$$\frac{\partial e_N}{\partial c_k} = \frac{\partial}{\partial c_k} \int_a^b (f(x) - \sum_{n=1}^{N} c_n f_n(x))^2 dx$$

$$= \int_a^b 2 (f(x) - \sum_{n=1}^{N} c_n f_n(x)) f_k(x) dx$$

$$= 2 \langle f, f_k \rangle - 2 \sum_{n=1}^{N} c_n \langle f_n, f_k \rangle$$

$$= 2 \langle f, f_k \rangle - 2 \sum_{n=1}^{N} c_n \|f_k\|^2$$

Vanishes only if

$$c_k = \frac{\langle f, f_k \rangle}{\|f_k\|^2}$$
So now we know the choice

\[ S_N(x) = \sum_{n=1}^{N} c_n f_n(x) \quad \text{where} \quad c_n = \frac{\langle f, f_n \rangle}{\|f_n\|^2} \]

minimizes the \( L^2 \) error \( e_N = \|f - S_N\|^2 \).

The next question is how big is this error?

\[ e_N = \|f - S_N\|^2 \]

\[ = \langle f - S_N, f - S_N \rangle \]

\[ = \langle f, f \rangle - 2 \langle f, S_N \rangle + \langle S_N, S_N \rangle \]

\[ = \|f\|^2 - 2 \langle f, \sum_{n=1}^{N} c_n f_n \rangle + \sum_{m=1}^{N} \sum_{n=1}^{N} \langle c_m f_m, c_n f_n \rangle \quad \text{orthog.} \]

\[ = \|f\|^2 - 2 \sum_{n=1}^{N} c_n \langle f, f_n \rangle + \sum_{n=1}^{N} c_n^2 \|f_n\|^2 \quad \text{orthog.} \]

\[ = \|f\|^2 - 2 \sum_{n=1}^{N} c_n^2 \|f_n\|^2 + \sum_{n=1}^{N} c_n^2 \|f_n\|^2 \]

So that

\[ e_N = \|f\|^2 - \sum_{n=1}^{N} c_n^2 \|f_n\|^2 \]

If additionally \( \{f_n\} \) are normalized, i.e. \( \|f_n\| = 1 \)

\[ e_N = \|f\|^2 - \sum_{n=1}^{N} c_n^2 \]
Parseval's Equality

If \( \{f_n\} \) is an orthonormal (normalized orthogonal) sequence then we proved

\[
(1) \quad e_N = \left\| \int f(x) - \sum_{n=1}^{N} c_n f_n(x) \right\|^2 = \|f\|^2 - \sum_{n=1}^{N} c_n^2
\]

for any \( N \) if the Fourier coefficients

\[
c_n = \langle f, f_n \rangle, \quad \|f_n\| = 1.
\]

If \( e_N \to 0 \) as \( N \to \infty \) (more terms included) then eqn (1) above implies

\[
(2) \quad \|f\|^2 = \sum_{n=1}^{\infty} c_n^2 \quad \{f_n\} \text{ orthonormal}
\]

which is known as Parseval's Equality.

In certain physical settings \( c_n^2 \) is the energy of the \( n \)-th "mode" and \( \|f\|^2 \) is the total energy.

Equation (2) can be used to derive formulae for certain infinite sums.
EXAMPLE
Previously we showed that
\[ f_n(x) = \sin(nx) \quad \text{and} \quad \|f_n\| = \sqrt{\frac{n}{\pi}} \]
form an orthogonal set on \( L^2[0, \pi] \)
Let
\[ \hat{f}_n = \sqrt{\frac{2}{\pi}} f_n \quad n \geq 1 \]
so that \( \{\hat{f}_n\} \) is an orthonormal set.

Assuming that
\[ f(x) = x = \sum_{n=1}^{\infty} c_n \hat{f}_n(x) \]
for some \( c_n \), derive a formula for the sum
\[ I = \sum_{n=1}^{\infty} c_n^2 \]

First compute the Fourier Coefficients
\[ c_n = \langle f, \hat{f}_n \rangle = \sqrt{\frac{2}{\pi}} \int_{0}^{\pi} x \sin(nx) dx \]
\[ c_n = \sqrt{\frac{2}{\pi}} \left( \frac{\sin(nx) - nx \cos(nx)}{n^2} \right) \bigg|_{0}^{\pi} \]
\[ c_n = \sqrt{\frac{2}{\pi}} \left( -\frac{\pi \cos(n\pi)}{n} \right) \]
\[ (2) \quad c_n = \sqrt{2\pi} \frac{(-1)^{n+1}}{n} \]
These are relative to the normalized basis!
Parseval's Equality

$$\| f \|_2^2 = \sum_{n=1}^{\infty} c_n^2$$

Given $f(x) = x$ and the result (2) this becomes

$$\int_0^\pi x^2 dx = \sum_{n=1}^{\infty} \left( \sqrt{2\pi} \frac{(-1)^{n+1}}{n} \right)^2$$

$$I = \frac{1}{3} \pi^3 = 2\pi \sum_{n=1}^{\infty} \frac{1}{n^2}$$

From which

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$