# Partial Differential Equations Introduction

Seek solns of differential equations involving two or more independent variables. For example solutions of

$$(1) \qquad u_{tt} = u_{xx} \qquad\qquad u = u(x,t)$$

Like ordinary differential equations, solutions are not unique. For (1) above

$$u(x,t) = f(x+t) + g(x-t)$$

is a solution for any sufficienty smooth functions $f(z)$, $g(z)$ of one argument, i.e.

$$u_{xx} = f''(x+t) + g''(x-t)$$

$$u_{tt} = f''(x+t) + g''(x-t)$$

Thus

$$\sin(x+t) \qquad\qquad e^{x-t} \qquad\qquad \tfrac{1}{2}(x+t)^3$$

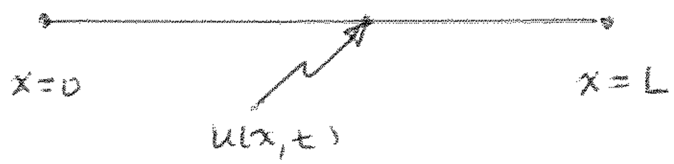
are all solutions.

Physics, Chemistry and Biology are replete with Partial Differential Equations (PDE).

**EXAMPLE** Heat Equation (1-D)

$$u_t = k\, u_{xx} \qquad x \in [0, L],\ t > 0$$

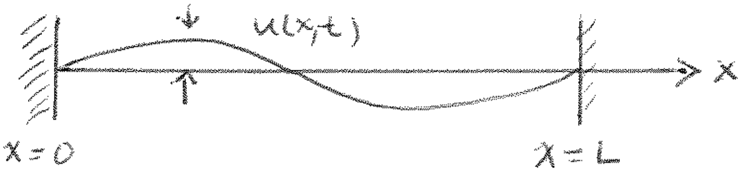
Here $u(x,t)$ is the temperature of a rod at position $x$, time $t$.




**EXAMPLE** Wave Equation (1-D)

$$u_{tt} = c^2 u_{xx} \qquad x \in [0, L],\ t > 0$$

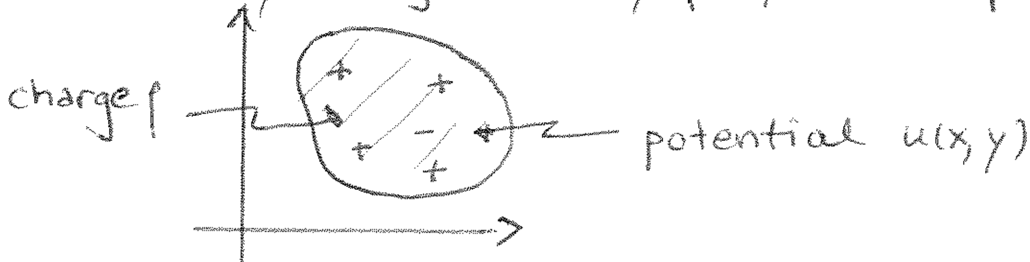
Here $u(x,t)$ is the displacement of a vibrating string




**EXAMPLE** Poisson's Equation (2-D)

$$u_{xx} + u_{yy} = \rho(x,y) \qquad (x,y) \in R$$

Here $u(x,y)$ is the electrostatic potential induced by charge density $\rho(x,y)$ on a plate



EXAMPLE Euler Tricomi Equation (Mixed Type)

$$u_{xx} = x\, u_{yy}$$

Here $u(x,y)$ is (approximately) the $x$ component of steady fluid velocity (near the object surface) in the transonic regime (near speed of sound)

EXAMPLE Schrödinger's Equation (Quantum)

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(\vec{r})\,\psi$$

where $\psi(\vec{r},t)$ is the pobabilty density function of a particle in the presence of potential $V(\vec{r})$, $\vec{r} = (x,y,z)$ is position.

$$\nabla^2 \psi = \psi_{xx} + \psi_{yy} + \psi_{zz} \qquad \text{cartesian}$$

$\hbar$ is Planck's constant, $m$ is mass.

For time independent case

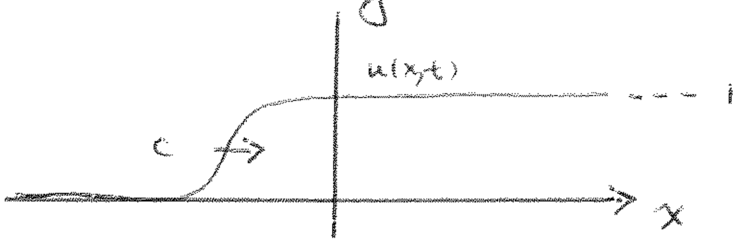
$$P = \iiint_R |\psi(x,y,z)|\, dV$$

is the probability of finding particle in region $R \subset \mathbb{R}^3$.

EXAMPLE **Fisher's Equation**

$$u_t = D u_{xx} + u(1-u) \qquad x \in \mathbb{R}.$$

where $D > 0$ constant. Here $u(x,t)$ is the population of a (noncompeting) organism at position $x$, time $t$. Organism motion is random. Equation admitts traveling wave solutions




extinction wave

EXAMPLE **Korteweg-de Vries Equation**

$$u_t + u u_x - \delta u_{xxx} = 0 \qquad x \in \mathbb{R}$$

where $u(x,t)$ is the spatial derivative of the displacement of a lattice of particles connected by nonlinear springs.

EXAMPLE **Eikonal equation**

$$u_x^2 + u_y^2 + u_z^2 = F(x)$$

Level sets $u(x,y,z) = k$ are wavefronts (of light) at some fixed $t$. Here $F(x) = c^{-2}$ where $c(x)$ is the local light speed of the medium.

EXAMPLE Radially Symmetric Wave Equation

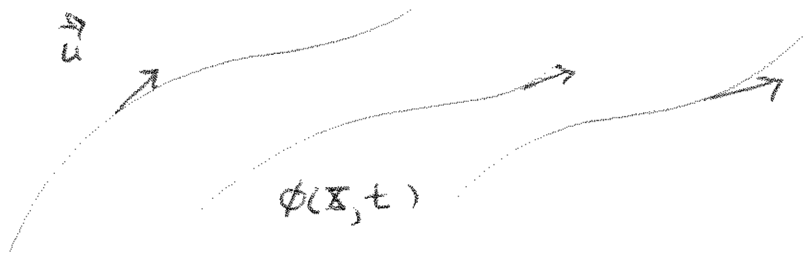
$$u_{tt} = c^2 \left( u_{rr} + \frac{2}{r} u_r \right) \qquad r > 0$$

Here $u(r,t)$ could be radial acoustic pressure waves emanating from a point source

EXAMPLE Advection (pure)

$$\phi_t + (u_1 \phi)_x + (u_2 \phi)_y + (u_3 \phi)_z = 0$$

Here $\phi(x,y,z,t)$ is the local concentration of a substance at position $\underline{X} = (x,y,z)$ and time $t$ in a fluid of velocity $\vec{u} = (u_1, u_2, u_3)$. Assumed absence of reactions and diffusion processes




The constant velocity 1-D version

$$\phi_t + c\,\phi_x = 0 \qquad c \in \mathbb{R} \text{ const.}$$

has the general soln

$$\phi(x,t) = f(x - ct)$$

for any $f(z) \in C^1(\mathbb{R})$.

# Second Order Partial Differential Equation

(1) $\quad G(u_{xx}, u_{xt}, u_{tt}, u_x, u_t, u, x, t) = 0$

A solution $u(x,t)$ is typically valid for some region $(x,t) \in R \subset \mathbb{R}^2$.

In some instances (1) can be written

(2) $\quad Lu = f(x,t) \qquad (x,t) \in R$

where

(3) $\quad Lu = a u_{tt} + b u_{xt} + c u_{xx} + d u_t + e u_x + g u$

If any of $a-g$ depend on $u$, the 2nd order PDE (2) is said to be quasilinear. If $a-g$ depend only on $(x,t)$ then (3) defines a linear operator:

$$L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2)$$

for any constants $c_k$. Additionally if $f(x,t) = 0$ in (2) the PDE is homogeneous

EXAMPLES

$u_{tt} + x u_x = 0$   2nd order, linear, homog.

$u_{tt} + u u_x = 0$   2nd order, quasilinear

$u_{xx} + u_{tt} = xt$   2nd order, linear, nonhomog.

## General Solution Examples

**EX** $\quad u_{tx} = tx$

$$u_t = \frac{1}{2} t x^2 + f(t)$$

where $f(t)$ is some arbitrary function of $t$. Integrate in $t$

$$u(x,t) = \frac{1}{4} t^2 x^2 + F(t) + G(x)$$

where $F, G$ are arbitrary.

**EX** $\quad u_{xt} + \frac{2}{x} u_t = t$

Note that

$$\frac{\partial}{\partial t}\left( u_x + \frac{2}{x} u \right) = t$$

$$u_x + \frac{2}{x} u = \frac{1}{2} t^2 + f(x)$$

where $f(x)$ is arbitrary. Is first order linear for $u$ (in $x$). Integrating factor $\mu(x) = \exp(+\int \frac{2}{x} dx) = x^2 \Rightarrow$

$$(\mu u)_x = \frac{1}{2} t^2 x^2 + x^2 f(x)$$

$$x^2 u = \frac{1}{6} t^2 x^3 + \int x^2 f(x) dx + g(t)$$

Consider $f, g$ arbitrary

$$u(x,t) = \frac{1}{6} t^2 x + F(x) + \frac{G(t)}{x^2}$$

for arbitrary $F, G$.

<u>EXAMPLE</u> $u u_t = xt$ lrst order nonlinear

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}u^2\right) = xt$$

Integrate in $t$. For some arbitrary function $f(x)$

$$u^2 = xt^2 + f(x)$$

$$u = \pm\sqrt{xt^2 + f(x)}$$

<u>EXAMPLE</u> Advection, nonhomogeneous

$$u_t + c\,u_x = f(x,t) \qquad c \in \mathbb{R}$$

Let $z = x - ct$ and $u(x,t) = v(z,t)$ then $v(z,t)$ satisfies

$$\left(v_z \frac{\partial z}{\partial t} + v_t\right) + c\,v_z \frac{\partial z}{\partial x} = f(z+ct, t)$$

Since $z_t = -c$, $z_x = 1$ this simplifies

$$v_t(z,t) = f(z+ct, t)$$

Integrate in $t$ ($z$-fixed)

$$v(z,t) = \int_0^t f(z + c\lambda, \lambda)\,d\lambda + g(z)$$

where $g(z)$ arbitrary. Thus

$$u(x,t) = \int_0^t f(x + c\lambda - ct, \lambda)\,d\lambda + g(x-ct)$$

## Linear Superposition

Recall that a partial differential operator $L$ is linear if

$$L(c_1 u_1 + c_2 u_2) = c_1 L(u_1) + c_2 L(u_2)$$

for all constants $c_K$ and functions $u_K(x,t)$.

**EX** Heat Equation is linear

$$L(u) \equiv u_t - u_{xx}$$

$$L(c_1 u_1 + c_2 u_2) = (c_1 u_1 + c_2 u_2)_t - (c_1 u_1 + c_2 u_2)_{xx}$$

$$= c_1 \frac{\partial u_1}{\partial t} - c_1 \frac{\partial^2 u_1}{\partial x^2} + c_2 \frac{\partial u_2}{\partial t} - c_2 \frac{\partial^2 u_2}{\partial x^2}$$

$$= c_1 L(u_1) + c_2 L(u_2)$$

**EX** A nonlinear PDE operator

$$L(u) = u_t + (u_x)^2$$

In particular for $c_1 = c_2 = 1$

$$L(u_1 + u_2) = (u_1 + u_2)_t + \left(\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial x}\right)^2$$

$$= \frac{\partial u_1}{\partial t} + \left(\frac{\partial u_1}{\partial x}\right)^2 + \frac{\partial u_2}{\partial t} + \left(\frac{\partial u_2}{\partial x}\right)^2 + 2\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial x}$$

$$= L(u_1) + L(u_2) + 2\frac{\partial u_1}{\partial x}\frac{\partial u_2}{\partial x}$$

$$\neq L(u_1) + L(u_2)$$

A consequence of linearity is that linear combinations of solutions of the homogeneous problem $Lu = 0$ are also solns.

$$\boxed{\begin{array}{l}\text{If } \; Lu_1 = 0 \text{ and } Lu_2 = 0 \text{ the } Lu = 0 \\ \text{if } \; u = c_1 u_1 + c_2 u_2\end{array}}$$

This is called the principal of linear superposition.

<u>EXAMPLE</u> <u>Advection equation</u>

Two solutions of

$$Lu \equiv u_t + c u_x = 0$$

are $u_1 = (x-ct)^2$ and $\sin(x-ct)$. Then, since $L$ is linear,

$$u(x,t) = 17(x-ct)^2 + \frac{1}{4}\sin(x-ct)$$

is also a solution of $Lu = 0$.

## Linear Superposition (Continuum Version)

Suppose $u(x,t;\zeta)$ is a solution of

$$(1) \qquad Lu = 0 \qquad \zeta \in I$$

for all $\zeta \in I$ (some interval) and that $L$ is a linear partial differential operator.

By linear superposition

$$(2) \qquad u_N(x,t) = \sum_{i=1}^{N} c_i \, u(x,t;\zeta_i)$$

is also a soln of the homog. problem $Lu_N = 0$.

Equation (2) looks very much like a Riemann sum. Indeed, one can show that if $c(\zeta)$ is a sufficienty smooth function (with certain decay properties) then

$$(3) \qquad \bar{u}(x,t) = \int_I c(\zeta)\, u(x,t,\zeta)\, d\zeta$$

is a homogeneous soln, i.e. $L\bar{u} = 0$

$$L\bar{u} = L\left(\int_I c(\zeta)\, u(x,t,\zeta)\, d\zeta\right)$$

validity depends on $c(\zeta)$

$$= \int_I c(\zeta)\, \cancel{Lu}\, d\zeta$$

$$= 0$$

EXAMPLE Heat Equation (IVP)

$$Lu \equiv u_t - k u_{xx} \qquad x \in \mathbb{R}, \quad t > 0$$

One can directly verify that

$$(1) \qquad u(x,t,\xi) = \frac{1}{\sqrt{4\pi k t}} \exp\left(\frac{-(x-\xi)^2}{4kt}\right)$$

is a solution of $Lu = 0$ for all $\xi \in \mathbb{R}$.

Thus (for $f(\xi)$ continuous and bounded)

$$\bar{u}(x,t) = \int_{\mathbb{R}} f(\xi)\, u(x,t,\xi)\, d\xi$$

$$(2) \qquad \bar{u}(x,t) = \int_{-\infty}^{\infty} f(\xi) \frac{1}{\sqrt{4\pi k t}} \exp\left(\frac{-(x-\xi)^2}{4kt}\right) d\xi$$

is a solution of $\bar{u}_t - k\bar{u}_{xx} = 0$. Here the function $f(\xi)$ is arbitrary.

We now show that $\bar{u}(x,t)$ solves the Initial Value Problem (IVP)

$$(3) \qquad \bar{u}_t - k\bar{u}_{xx} = 0$$

$$(4) \qquad \bar{u}(x,0^+) = f(x)$$

Technically $\bar{u}$ is undefined at $t=0$ but it is in the limit $t \to 0^+$.

Make the change of variables

$$r = \frac{\xi - x}{\sqrt{4kt}}$$

$$\xi = x + \sqrt{4kt}\, r$$

Since (for fixed $x, t$) $\xi \in \mathbb{R} \Rightarrow r \in \mathbb{R}$ and the integral (2) becomes

$$\bar{u}(x,t) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} f(x + \sqrt{4kt}\, r)\, e^{-r^2}\, dr$$

In the limit $t \to 0^+$

$$\bar{u}(x, 0^+) = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} f(x)\, e^{-r^2}\, dr$$

$$\bar{u}(x, 0^+) = f(x) \left\{ \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-r^2}\, dr \right\} \text{ integral is } \underline{1} \text{ (from tables)}$$

$$\bar{u}(x, 0^+) = f(x)$$

Some concluding remarks.

In a distributional (weak) sense

(1) $$g(x,t,\xi) = \frac{1}{\sqrt{4\pi k t}} \exp\left(-\frac{(x-\xi)^2}{4kt}\right)$$

is the solution of

$$Lg = g_t - k g_{xx} = \delta(x-\xi)\,\delta(t) \qquad x \in \mathbb{R}$$

We do not prove that here. In parallel to Sturm Liouville theory, $g(x,t,\xi)$ is the <u>fundamental solution</u> of the heat equation.

Moreover, $g(x,t,\xi)$ is the <u>Green's function</u> for the heat equation IVP in the sense that the solution of

(2) $$u_t - k u_{xx} = 0$$

(3) $$u(x,0) = f(x)$$

is given by

$$u(x,t) = \int_{\mathbb{R}} f(\xi)\, g(x,t,\xi)\, d\xi$$