

Eigenfunctions of the Laplacian

An eigenvalue λ of $L \equiv -\nabla^2$ is any λ such that

$$(1) \quad -\nabla^2 u = \lambda u \quad \vec{x} \in \Omega$$

$$(2) \quad B(u) \equiv \alpha u + \beta \frac{\partial u}{\partial n} = 0 \quad \vec{x} \in \partial\Omega$$

has a nontrivial solution $u_\lambda(\vec{x})$. Here $\lambda \neq 0$ (else (1)-(2) is ill posed) and $u_\lambda(\vec{x})$ is an eigenfunction. Note the boundary conditions are homogeneous (zero).

Inner Product for $L^2(\Omega)$

$$\langle u, v \rangle \equiv \int_{\Omega} u(\vec{x}) v(\vec{x}) d\vec{x}$$

defines an inner product for square integrable functions on $\Omega \subset \mathbb{R}^n$

Eigenvalue Properties

$$(a) \quad \lambda \in \mathbb{R}$$

$$(b) \quad \langle u_{\lambda_1}, u_{\lambda_2} \rangle = 0 \quad \lambda_1 \neq \lambda_2 \quad (\text{orthogonality})$$

(c) Set of $\{u_\lambda(\vec{x})\}$ is complete in sense that if $f \in L^2(\Omega)$ there are c_λ s.t.

$$f(\vec{x}) = \sum_{\lambda} c_{\lambda} u_{\lambda}(\vec{x})$$

Typically expansions are double/triple sums.

Orthogonality of eigenfunctions (Proof example)

We prove here that eigenfunctions of

$$(1) \quad -\nabla^2 u = \lambda u \quad \vec{x} \in \Omega$$

$$(2) \quad u = 0 \quad \vec{x} \in \partial\Omega$$

having distinct eigenvalues are orthogonal

Pf/ Let $\lambda_1 \neq \lambda_2$ and u_k be nontrivial solutions of

$$(3) \quad -\nabla^2 u_k = \lambda_k u_k \quad u_k|_{\partial\Omega} = 0$$

Green's Second identity for ϕ, ψ

$$(4) \quad \int_{\Omega} (\phi \nabla^2 \psi - \psi \nabla^2 \phi) d\vec{x} = \int_{\partial\Omega} (\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n}) dS$$

Let $\phi = u_1$ and $\psi = u_2$ in (4) noting u_k vanishes on boundary $\partial\Omega$ so right side is zero.

$$(5) \quad \int_{\Omega} (u_1 \nabla^2 u_2 - u_2 \nabla^2 u_1) d\vec{x} = 0$$

using (3) in (5) we find

$$(\lambda_1 - \lambda_2) \int_{\Omega} u_1 u_2 d\vec{x} = (\lambda_1 - \lambda_2) \langle u_1, u_2 \rangle = 0$$

from which $\langle u_1, u_2 \rangle = 0$ if $\lambda_1 \neq \lambda_2$ \square

NOTE Green's 2nd identity (4) follows from the Divergence Thm with $\vec{F} = \phi \vec{\nabla} \psi - \psi \vec{\nabla} \phi$.

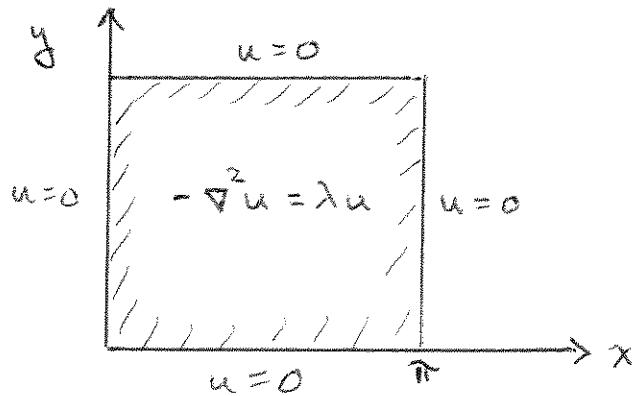
EXAMPLE

Laplacian on $\Omega = [0, \pi]^2$

$$(1) \quad - (u_{xx} + u_{yy}) = \lambda u$$

$$(2) \quad u = 0 \quad (x, y) \in \partial\Omega$$

Note that u vanishes on edge of square.



We use a technique known as Separation of Variables to find the eigenfunctions

Assume there are functions $\bar{X}(x)$ and $\bar{Y}(y)$ s.t.

$$u(x, y) = \bar{X}(x) \bar{Y}(y)$$

is a nontrivial soln of (1)-(2) for some $\lambda \in \mathbb{R}$.
Then

$$(3) \quad -\bar{X}'' \bar{Y} - \bar{X} \bar{Y}'' = \lambda \bar{X} \bar{Y}$$

$$(4) \quad \bar{X}(0) = 0 \quad \bar{X}(\pi) = 0$$

$$(5) \quad \bar{Y}(0) = 0 \quad \bar{Y}(\pi) = 0$$

Divide (3) by $\bar{X}\bar{Y}$ and rearrange

$$(6) \quad -\frac{\bar{X}''(x)}{\bar{X}(x)} = \frac{\bar{Y}''(y) + \lambda \bar{Y}(y)}{\bar{Y}(y)} = \mu$$

fn of x
 only

 fn of y
 only

Since x, y are independent variables
 the only way a function of x can
 equal a function of y is that they
 are both constant functions. This
 is indicated by the constant μ in (6).

Given (6) and the boundary conditions (4)-(5)
 $\bar{X}(x)$ and $\bar{Y}(y)$ must be nontrivial solutions
 of

$$(7) \quad \bar{X}'' + \mu \bar{X} = 0 \quad \bar{X}(0) = \bar{X}(\pi) = 0$$

$$(8) \quad \bar{Y}'' + (\lambda - \mu) \bar{Y} = 0 \quad \bar{Y}(0) = \bar{Y}(\pi) = 0$$

The solution of (7) is

$$\bar{X}_m(x) = \sin(mx) \quad \mu_m = m^2 \quad m = 1, 2, 3, \dots$$

Then (8) becomes

$$\bar{Y}'' + (\lambda - m^2) \bar{Y} = 0 \quad \bar{Y}(0) = \bar{Y}(\pi) = 0$$

This has nontrivial solutions only if $\lambda - m^2 = n^2$
 for some integer $n = 1, 2, 3, \dots$. Hence

$$\lambda_{mn} = m^2 + n^2$$

Conclude eigenfunction/eigenvalue pairs are

$$(9) \quad u_{nm}(x, y) = \sin(mx) \sin(ny)$$

$$(10) \quad \lambda_{mn} = m^2 + n^2$$

The eigenfunctions in (9) are not normalized but can be. Indeed direct calculations reveal

$$\|u_{nm}\|^2 = \int_0^\pi \int_0^\pi \sin^2(mx) \sin^2(ny) dy dx$$

$$\|u_{nm}\|^2 = \frac{\pi^2}{4}$$

So that the normalized eigenfunctions are

$$\hat{u}_{nm}(x, y) = \frac{2}{\pi} \sin(mx) \sin(ny)$$

EXAMPLE Find a series solution of

$$(1) \quad -\nabla^2 \phi = f(x, y) \quad (x, y) \in [0, \pi]^2$$

$$(2) \quad \phi = 0 \quad (x, y) \in \partial\Omega$$

using the eigenfunctions of $-\nabla^2 u$.

Let

$$(3) \quad \phi = \sum_n \sum_m \phi_{mn} \hat{u}_{mn}(x, y)$$

$$(4) \quad f = \sum_n \sum_m f_{mn} \hat{u}_{mn}(x, y) \quad f_{mn} = \langle f, \hat{u}_{mn} \rangle$$

Note ϕ satisfies the boundary conditions since the \hat{u}_{mn} do. Substitute (3)-(4) into (1) using $-\nabla^2 \hat{u}_{mn} = \lambda_{mn} \hat{u}_{mn}$.

$$\sum_n \sum_m \lambda_{mn} \phi_{mn} \hat{u}_{mn}(x, y) = \sum_n \sum_m f_{mn} \hat{u}_{mn}(x, y)$$

Orthogonality of \hat{u}_{mn} implies series coeff. equal \Rightarrow

$$(5) \quad \phi_{mn} = \frac{f_{mn}}{\lambda_{mn}} = \frac{f_{mn}}{m^2 + n^2}$$

We know $\lambda_{mn} = m^2 + n^2$. Were $f(x, y) = xy$ we could calculate

$$f_{mn} = \frac{2}{\pi} \int_0^\pi \int_0^\pi xy \sin(mx) \sin(ny) dy dx$$

$$f_{mn} = \frac{2\pi (-1)^{m+n}}{mn}$$

When used in (5), ϕ_{mn} known for that f .

EXAMPLELaplacian on unit circle

In polar coordinates one can show

$$\nabla^2 u = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}$$

Here we seek the eigenfunctions (bounded) of the Laplacian operator having homogeneous Dirichlet boundary conditions. Such $u(r, \theta)$ are solns of

$$(1) \quad \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} = -\lambda u$$

$$(2) \quad u(1, \theta) = 0$$

where $\Omega = \{(r, \theta) : r \in [0, 1], \theta \in [0, 2\pi)\}$.

Here we sketch out how the method of separation of variables can be used to find the eigenfunctions.

Assume

$$u(r, \theta) = R(r) \Theta(\theta)$$

Using this in (1) we find

$$(3) \quad \frac{r(rR')' + \lambda r^2 R}{R} + \frac{\Theta''}{\Theta} = 0$$

Since $u(r, \theta)$ must be 2π -periodic in θ

$$\Theta_m(\theta) = a_m \cos(m\theta) + b_m \sin(m\theta)$$

for $m=0, 1, 2, \dots$ noting $\Theta_0(\theta) = a_0$ a constant.

Then (3) becomes

$$(4) \quad r(R')' + (\lambda r^2 - m^2)R = 0$$

$$(5) \quad R(1) = 0$$

The boundary condition (5) assures $u(1,0) = 0$

The general solution of (4) involves Bessel functions of the first and second kind

$$R(r) = A J_m(\sqrt{\lambda}r) + B Y_m(\sqrt{\lambda}r)$$

Only $J_m(z)$ is bounded as $z \rightarrow 0^+$ so

$$R(r) = A J_m(\sqrt{\lambda}r)$$

The boundary condition implies the eigenvalues λ are roots of

$$(6) \quad J_m(\sqrt{\lambda}) = 0$$

For each fixed m this has a countable set of roots, i.e.

$$(7) \quad J_m(\sqrt{\lambda_{mn}}) = 0 \quad m \geq 0, n \geq 1$$

This defines λ_{mn} . The eigenfunctions are

$$u_{mn}^{(1)}(r, \theta) = \cos(m\theta) J_m(\sqrt{\lambda_{mn}}r)$$

$$u_{mn}^{(2)}(r, \theta) = \sin(m\theta) J_m(\sqrt{\lambda_{mn}}r)$$

Since the eigenfunctions form a complete set, a smooth function $f(r, \theta)$ can be expanded:

$$f(r, \theta) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_{mn} u_{mn}^{(1)}(r, \theta) + b_{mn} u_{mn}^{(2)}(r, \theta)$$

The eigenfunctions are mutually orthogonal under the inner product

$$\langle u, v \rangle = \int_{\Omega} u(\vec{x}) v(\vec{x}) d\vec{x} = \int_0^{2\pi} \int_0^r u(r, \theta) v(r, \theta) r dr d\theta$$

noting the area element is $r dr d\theta$ in polar.

Thus, in particular

$$\langle f, u_{mn}^{(1)} \rangle = a_{mn} \|u_{mn}^{(1)}\|^2$$

can be used to compute a_{mn} given $f(r, \theta)$.

Heat Equation Series Solutions (Example)

$$(1) \quad u_t = D \nabla^2 u \quad \vec{x} \in \Omega, t > 0$$

$$(2) \quad u(\vec{x}, t) = 0 \quad \vec{x} \in \partial\Omega$$

$$(3) \quad u(\vec{x}, 0) = f(\vec{x}) \quad \vec{x} \in \Omega$$

Specifics will depend on the spatial domain Ω and the boundary conditions. Here we have Dirichlet boundary conditions but they may be mixed.

Let $u_\lambda(\vec{x})$ be the eigenfunctions satisfying

$$(4) \quad -\nabla^2 u_\lambda = \lambda u_\lambda \quad \vec{x} \in \Omega$$

$$(5) \quad u_\lambda(\vec{x}) = 0 \quad \vec{x} \in \partial\Omega$$

Previously we showed how to find such $u_\lambda(\vec{x})$ in the cases Ω a square and circle.

Given one knows $u_\lambda(\vec{x})$ the solution $u(\vec{x}, t)$ of (1)-(3) above will have a series expansion of the form

$$(6) \quad u(\vec{x}, t) = \sum_{\lambda} a_{\lambda}(t) u_{\lambda}(\vec{x})$$

noting the sum is a double sum if $\Omega \subset \mathbb{R}^2$

To find the Fourier Coefficients $a_{\lambda}(t)$ we substitute the expansion (6) in (1).

$$\sum_{\lambda} \frac{\partial}{\partial t} (a_{\lambda}(t) u_{\lambda}(\vec{x})) = D \sum_{\lambda} a_{\lambda}(t) \nabla^2 u_{\lambda}$$

Since u_{λ} are eigenfunctions of ∇^2 this becomes

$$\sum_{\lambda} a'_{\lambda}(t) u_{\lambda}(\vec{x}) = \sum_{\lambda} D a_{\lambda}(t) (-\lambda u_{\lambda})$$

or

$$\sum_{\lambda} \underbrace{(a'_{\lambda}(t) + \lambda D a_{\lambda}(t))}_{=0} u_{\lambda}(\vec{x}) = 0$$

Orthogonality of $u_{\lambda}(\vec{x})$ assures the indicated term vanishes so

$$a_{\lambda}(t) = a_{\lambda}(0) e^{-D\lambda t}$$

where $a_{\lambda}(0)$ is an unknown constant for each λ .
Thus (6) becomes

$$(7) \quad u(\vec{x}, t) = \sum_{\lambda} a_{\lambda}(0) e^{-D\lambda t} u_{\lambda}(\vec{x})$$

To find $a_{\lambda}(0)$ we use the initial condition (3)

$$f(\vec{x}) = \sum_{\lambda} a_{\lambda}(0) u_{\lambda}(\vec{x})$$

hence (by orthogonality of $u_{\lambda}(\vec{x})$)

$$(8) \quad a_{\lambda}(0) = \frac{\langle f, u_{\lambda} \rangle}{\|u_{\lambda}\|^2}$$

Collectively (7)-(8) define the soln of (1)-(3).

EXAMPLE

Heat Equation $\Omega = (0, \frac{\pi}{2})$

(1) $u_t = Du_{xx} \quad x \in \Omega, t > 0$

(2) $u_x(0, t) = u(\frac{\pi}{2}, t) = 0$

(3) $u(x, 0) = f(x)$

Here $\nabla^2 = \frac{d^2}{dx^2}$ is the one dimensional Laplacian.

Need eigenfunctions of

(4) $-\frac{d^2 u_\lambda}{dx^2} = \lambda u$

(5) $u'_\lambda(0) = 0 \quad u_\lambda(\frac{\pi}{2}) = 0$

The general solution of (4) is

$$u_\lambda(x) = A \cos(\mu x) + B \sin(\mu x) \quad \lambda = \mu^2$$

The boundary condition $u'_\lambda(0) = 0 \Rightarrow B = 0$ so

$$u_\lambda(x) = A \cos(\mu x)$$

Then

$$u_\lambda(\frac{\pi}{2}) = A \cos(\mu \frac{\pi}{2}) = 0$$

only if

$$\mu \cdot \frac{\pi}{2} = \frac{\pi}{2} + m\pi \quad m = 0, 1, 2, \dots$$

or that

(6)

$$\mu_m = 2m+1 \quad m \geq 0$$

$$u_m(x) = \cos(\mu_m x)$$

and $\lambda_m = \mu_m^2$

The eigenfunctions (6) are then used to solve the original problem

$$u(x, t) = \sum_{m=0}^{\infty} a_m(t) \cos(\mu_m x)$$

Substituting this into the PDE eqn (1)

$$\sum_{m=0}^{\infty} a_m'(t) \cos(\mu_m x) = \sum_{m=0}^{\infty} -D\mu_m^2 a_m(t) \cos(\mu_m x)$$

The orthogonality of $\{\cos(\mu_m x)\}$ then implies

$$a_m'(t) = -D\mu_m^2 a_m(t)$$

whose general solution is

$$a_m(t) = a_m(0) e^{-D\mu_m^2 t}$$

Then

$$(7) \quad u(x, t) = \sum_{m=0}^{\infty} a_m(0) e^{-D\mu_m^2 t} \cos(\mu_m x)$$

For a given initial condition $u(x, 0) = f(x)$

$$f(x) = \sum_{m=0}^{\infty} a_m(0) \cos(\mu_m x)$$

Since $\|\cos(\mu_m x)\|^2 = \frac{\pi}{4}$ we then find

$$a_m(0) = \frac{4}{\pi} \langle f, \cos(\mu_m x) \rangle = \frac{4}{\pi} \int_0^{\pi/2} f(x) \cos(\mu_m x) dx$$

Using these values completes the soln (7).

Wave Equation Series solutions (Example)

$$(1) \quad u_{tt} = c^2 \nabla^2 u \quad \vec{x} \in \Omega, t > 0$$

$$(2) \quad u(\vec{x}, 0) = f(\vec{x})$$

$$(3) \quad u_t(\vec{x}, 0) = g(\vec{x})$$

Here Ω is some bounded spatial domain on which we additionally stipulate the Dirichlet B.C.

$$(4) \quad u(\vec{x}, t) = 0 \quad \vec{x} \in \partial\Omega$$

As before we construct $u(\vec{x}, t)$ from a superposition of the eigenfns of the Laplace operator

$$(5) \quad u(\vec{x}, t) = \sum_{\lambda} \alpha_{\lambda}(t) u_{\lambda}(\vec{x})$$

Substitution of (5) and the orthogonality of $u_{\lambda}(\vec{x})$ then imply

$$\alpha_{\lambda}'' = -c^2 \lambda \alpha_{\lambda}$$

whose general solution is

$$\alpha_{\lambda}(t) = a_{\lambda} \cos(cpt) + b_{\lambda} \sin(cpt)$$

where $\lambda = p^2$ and a_{λ}, b_{λ} are constants.

Thus (5) becomes

$$(6) \quad u(\vec{x}, t) = \sum_{\lambda} (a_{\lambda} \cos(c\mu t) + b_{\lambda} \sin(c\mu t)) u_{\lambda}(\vec{x})$$

The unknown constants a_{λ} and b_{λ} are then found using the initial conditions (2)-(3). Using the expression above

$$(7) \quad u(\vec{x}, 0) = \sum_{\lambda} a_{\lambda} u_{\lambda}(\vec{x}) = f(\vec{x})$$

$$(8) \quad u_t(\vec{x}, 0) = \sum_{\lambda} b_{\lambda} c\mu u_{\lambda}(\vec{x}) = g(\vec{x})$$

Using the orthogonality of the eigenfns $u_{\lambda}(\vec{x})$ we find

$$a_{\lambda} = \frac{\langle u_{\lambda}, f \rangle}{\|u_{\lambda}\|^2} \quad b_{\lambda} = \frac{\langle u_{\lambda}, g \rangle}{c\mu \|u_{\lambda}\|^2}$$

where $\mu = \sqrt{\lambda}$ and λ is the eigenvalue of the Laplacian

$$-\nabla^2 u_{\lambda} = \lambda u_{\lambda} \quad u_{\lambda}|_{\partial\Omega} = 0$$

Again the inner product is

$$\langle u, v \rangle = \int_{\Omega} u(\vec{x}) v(\vec{x}) d\vec{x}$$

in general. If $\Omega = (a, b)$ is an interval

$$\langle u, v \rangle = \int_a^b u(x) v(x) dx.$$

EXAMPLE Wave Equation on $\Omega = (0, \pi)$

$$(1) \quad u_{tt} = c^2 u_{xx}$$

$$(2) \quad u(0, t) = u_x(0, t) = 0 \quad \text{B.C.}$$

$$(3) \quad u(x, 0) = f(x)$$

$$(4) \quad u_t(x, 0) = 0$$

Need eigenfunctions of Laplacian $L = -\frac{d^2}{dx^2}$

$$(5) \quad -u_{xx} = \lambda u \quad \lambda = \mu^2$$

$$(6) \quad u(0) = 0 = u_x(0)$$

The (nonnormalized) eigenfunctions and associated eigenvalues for this Sturm-Liouville problem are

$$u_m(x) = \sin(\mu_m x)$$

$$\mu_m = m + \frac{1}{2} \quad m = 0, 1, 2, \dots$$

Then the series solution $u(x, t)$ of (1)-(4) has the form

$$u(x, t) = \sum_{m=0}^{\infty} (a_m \cos(c\mu_m t) + b_m \sin(c\mu_m t)) \sin(\mu_m x)$$

It is easy to verify from this that

$$u_t(x, 0) = \sum_{m=0}^{\infty} c b_m \mu_m \sin(\mu_m x)$$

Since $u_t(x, 0) = 0$ we conclude $b_m = 0 \quad \forall m$.

Thus

$$(7) \quad u(x,t) = \sum_{m=0}^{\infty} a_m \cos(\omega_m t) \sin(\mu_m x)$$

At this stage we know $u(x,t)$ satisfies the PDE, both boundary conditions and the initial condition $u_t(x,0) = 0$. We use the other initial condition to determine the unknown constants a_m .

$$u(x,0) = f(x) = \sum_{m=0}^{\infty} a_m \sin(\mu_m x)$$

Since $\|\sin \mu_m x\|^2 = \frac{\pi}{2}$,

$$a_m = \frac{\langle f, \sin(\mu_m x) \rangle}{\|\sin \mu_m x\|^2}$$

$$(8) \quad a_m = \frac{2}{\pi} \int_0^\pi f(x) \sin(\mu_m x) dx$$

For a given $f(x)$, one computes (8) and uses it in (7) to find $u(x,t)$.

Specific example: $f(x) = \sin^2 x$

$$a_m = \frac{2}{\pi} \int_0^\pi (\sin^2 x) \sin(\mu_m x) dx$$

can be computed using tables.

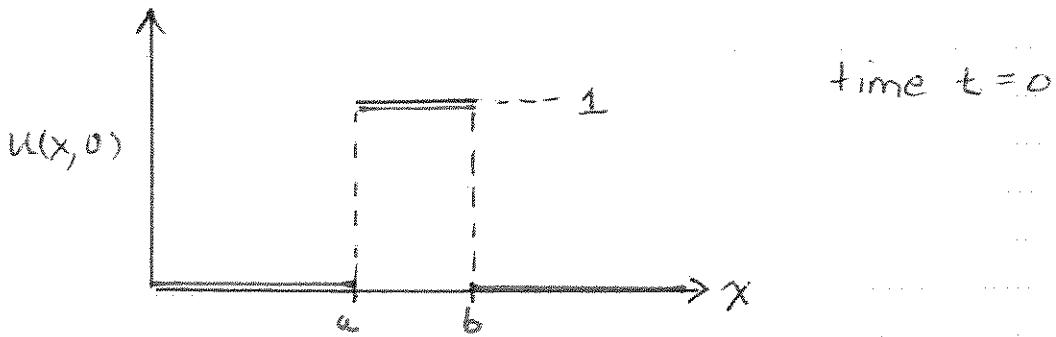
$$a_m = -\frac{32}{\pi} \frac{1}{(2m+1)(2m-3)(2m+5)}$$

and $u(x,t)$ is given explicitly as

$$u(x,t) = -\frac{32}{\pi} \sum_{m=0}^{\infty} \frac{\cos(\omega_m t) \sin(\omega_m x)}{(2m+1)(2m-3)(2m+5)}$$

Wave Propagation in $\Omega = (0, \pi)$

To better see wave propagation in this problem we include a truncated series approximation to $u(x,t)$

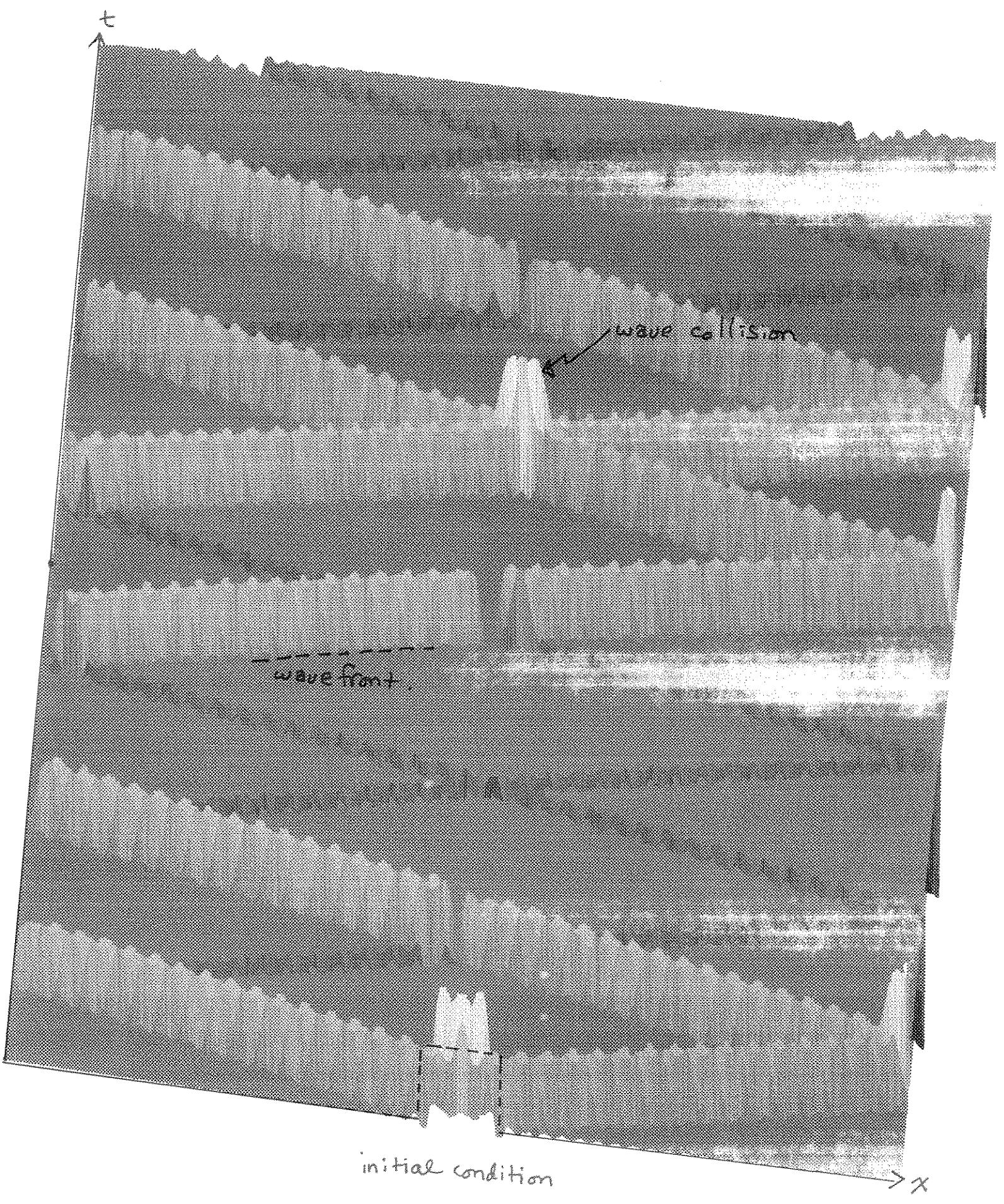


Here $u(x,0) = H(x-a) - H(x-b)$ and $u_t(x,0) = 0$

We pick the wavespeed $c = 1$, compute a_m for the initial condition above. Then

$$u(x,t) \approx S_N \equiv \sum_{m=0}^N a_m \cos(\omega_m t) \sin(\omega_m x)$$

On the next page we show an $N=100$ term approximation of $u(x,t)$. The wave behavior for $a=1.5$, $b=1.6$ is clearly evident.



Analytic expression of wave behavior.

For the previous problem the general series solution was found to be

$$u(x,t) = \sum_{m=0}^{\infty} a_m \cos(c\mu_m t) \sin(\mu_m x), \quad \mu_m = \frac{b}{2} + m$$

We use the identity

$$\cos(A) \sin(B) = \frac{1}{2} \sin(B+A) + \frac{1}{2} \sin(B-A)$$

For the series above $A = c\mu_m t$ and $B = \mu_m x$.

Thus one may write

$$u(x,t) = \frac{1}{2} \sum_{m=0}^{\infty} a_m \underbrace{\sin(\mu_m(x+ct))}_{\text{fn of } (x+ct)} + a_m \underbrace{\sin(\mu_m(x-ct))}_{\text{fn of } (x-ct)}$$

Thus the solution $u(x,t)$ is the superposition of left and right moving waves.