Perturbation Theory

Goal : Approximate solns of problems containing a small parameter $\varepsilon \ll 1$

(1) Algebraic Problems

$$f(x, \varepsilon) = 0 \qquad x \in \mathbb{R}^n$$

(2) Differential Equations (ordinary)

$$F(y'', y', y, t, \varepsilon) = 0$$

(3) Systems of Differential Eqns

$$\frac{dx}{dt} = f(x, \varepsilon) \qquad x \in \mathbb{R}^n$$

(4) Integrals

$$I(\varepsilon) = \int_{a(\varepsilon)}^{b(\varepsilon)} F(x, \varepsilon)\, dx$$

(5) Partial Differential Equations

$$u_t = \varepsilon D u_{xx} + f(u, \varepsilon) \qquad u \in \mathbb{R}^n$$

Introductory Examples

EXAMPLE Consider the quadratic

$$(1) \qquad f(x, \varepsilon) = x^2 + \varepsilon x - 1 = 0$$

where $0 < \varepsilon \ll 1$. Neglecting the term εx one would obtain the approximation

$$x \simeq \pm 1$$

Here we know the exact solns

$$(2) \qquad \bar{x}_{\pm}(\varepsilon) = -\tfrac{1}{2}\varepsilon \pm \tfrac{1}{2}(4+\varepsilon^2)^{1/2}$$

Does the term εx increase or decrease the value of the roots ± 1 when $\varepsilon = 0$?

The exact formula (2) is not terribly useful toward answering this question.

A Taylor series of $\bar{x}_{\pm}(\varepsilon)$ about $\varepsilon = 0$ yields

$$\bar{x}_{\pm}(\varepsilon) = \pm 1 - \tfrac{1}{2}\varepsilon + O(\varepsilon^2)$$

Clearly the roots decrease by $\tfrac{1}{2}\varepsilon$.

Even if one knows the exact soln it may not be useful.

EXAMPLE Consider the cubic

(1) $$f(x,\varepsilon) = x^3 + \varepsilon x - 1 = 0 \qquad 0 < \varepsilon \ll 1$$

Since $f_x = 3x^2 + \varepsilon > 0$, there is one real root, i.e $f(x,\varepsilon)$ increases in x.

Closed form solutions for cubics exist.

Here the sole real root is

$$\bar{x}(\varepsilon) = \Delta_+(\varepsilon) - \Delta_-(\varepsilon)$$

where

$$\Delta_{\pm}(\varepsilon) = \left(\pm \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\varepsilon^3}{27}} \right)^{1/3}$$

This formula is arguably too complicated to have practical importance/use.

However, the Taylor series is useful

$$\bar{x}(\varepsilon) = 1 - \varepsilon + \frac{1}{3}\varepsilon^3 + O(\varepsilon^4)$$

tells us, when ε is very small, the root decreases as $\varepsilon \uparrow$.

EXAMPLE Neglecting terms can be bad

Consider

$$(2) \quad f(x, \varepsilon) = \varepsilon x^2 + x - 1 = 0$$

neglecting the εx^2 term we arrive at

$$x \simeq 1$$

but (1) must have two roots. They are:

$$\bar{x}_{\pm}(\varepsilon) = \frac{\pm 1 + (1+4\varepsilon)^{1/2}}{2\varepsilon}$$

One can show

$$\bar{x}_{-}(\varepsilon) = 1 - \varepsilon + 2\varepsilon^2 + O(\varepsilon^3)$$

$$\bar{x}_{+}(\varepsilon) = -\frac{1}{\varepsilon} - 1 + \varepsilon - 2\varepsilon^2 + O(\varepsilon^3)$$

while its true $\bar{x}_{-}(\varepsilon) \simeq 1$ is a good approximation of one root, neglecting εx^2 we missed a large one

$$\varepsilon \bar{x}_{+}^{2} = O\left(\frac{1}{\varepsilon}\right)$$

EXAMPLE Neglecting terms can be really bad

$$f(x, \varepsilon) = \varepsilon x^2 - 1 = 0$$

Setting $\varepsilon = 0$ yields the contradictory statement

$$-1 = 0$$