Regular Perturbations (Algebraic)

Let $f : \mathbb{R} \times I \to \mathbb{R}$, $\varepsilon \in I$. Seek a soln(s) x of

$$(1) \qquad f(x, \varepsilon) = 0 \qquad x \in \mathbb{R}.$$

of the form

$$(2) \qquad x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$$

The expansion (2) is said to be <u>regular in ε</u>.

Eqn (2) is an assumption that the solution $x(\varepsilon)$ is smooth in ε and as a consequence has a Taylor series expansion, i.e.,

$$x(\varepsilon) = x(0) + \varepsilon x'(0) + \frac{1}{2!} \varepsilon^2 x''(0) + \cdots$$

whether (2) is a valid assumption depends on f.

One theorem addressing this: is the Implicit Function Thm.

<u>Theorem</u> Let $f : \mathbb{R} \times I \to \mathbb{R}$ and assume f and all its partial derivatives are smooth. Then if

$$(a) \qquad f(\bar{x}, 0) = 0$$
$$(b) \qquad f_x(\bar{x}, 0) \neq 0$$

then $\exists$ a smooth fn $x(\varepsilon)$ with $x(0) = \bar{x}$ such that $f(x(\varepsilon), \varepsilon) = 0$ for all ε in some neighbourhood of $\varepsilon = 0$.

EXAMPLE $\boxed{x^3 + \varepsilon x - 1 = 0 \;, \quad 0 < \varepsilon \ll 1}$

Assume

(1) $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$

Use (1) in $f(x, \varepsilon) = 0$

$$(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^3 + \varepsilon(x_0 + \varepsilon x_1 + \cdots) - 1 = 0$$

Expand and collect in powers of ε.

$$(x_0^3 + 3x_0^2 x_1 \varepsilon + (3x_0 x_1^2 + 3x_0^2 x_2)\varepsilon^2) + \varepsilon(x_0 + \varepsilon x_1 + \cdots) - 1 = 0$$

Arrive at a set of eqns which can be solved.

$O(1)$ $\quad \underline{x_0}^3 - 1 = 0$ $\quad$ solve for x_0

$O(\varepsilon)$ $\quad 3x_0^2 \underline{x_1} + x_0 = 0$ $\quad$ then x_1

$O(\varepsilon^2)$ $\quad 3x_0^2 \underline{x_2} + 3x_0 x_1^2 + x_1 = 0$ $\quad$ then x_2

Solving sequentially

$$x_0 = 1$$

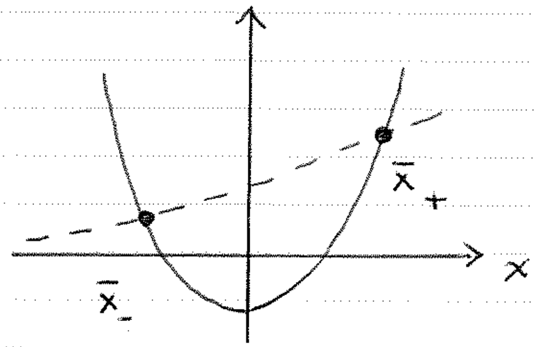
$$x_1 = -\frac{1}{3}$$

$$x_2 = 0$$

To conclude

$$x(\varepsilon) = 1 - \frac{1}{3}\varepsilon + O(\varepsilon^3)$$

EXAMPLE $\boxed{x^2 - 1 - \varepsilon e^x = 0, \quad 0 < \varepsilon \ll 1}$



$x^2 - 1 = \varepsilon e^x$

intersection of curves

clearly two roots

Seek a regular approximation for both roots

(1) $$\bar{x}(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3)$$

Need an expansion for $e^{\bar{x}(\varepsilon)}$. First note

(2) $$e^{x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots} = e^{x_0} e^{\varepsilon(x_1 + \varepsilon x_2 + \cdots)}$$

Letting $z = \varepsilon(x_1 + \varepsilon x_2 + \cdots)$ which is small

$$e^z = 1 + z + \frac{1}{2} z^2 + \cdots$$

$$e^z = 1 + (\varepsilon x_1 + \varepsilon^2 x_2 + \cdot) + \frac{1}{2}(\varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^2$$

(3) $$e^z = 1 + \varepsilon x_1 + (x_2 + \tfrac{1}{2} x_1^2)\varepsilon^2 + O(\varepsilon^3)$$

Using (3) in (2) we find

(4) $$e^{\bar{x}(\varepsilon)} = e^{x_0} + \varepsilon x_1 e^{x_0} + (x_2 + \tfrac{1}{2} x_1^2) e^{x_0} \varepsilon^2 + \cdots$$

Now use (4) and (1) in $f(x, \varepsilon) = 0$, expand and collect powers of ε.

$$(x_0+\varepsilon x_1+\varepsilon^2 x_2+\cdot)^2 \quad -1 - \varepsilon(e^{x_0}+\varepsilon x_1 e^{x_0}+O(\varepsilon^2)) = 0$$

$$\underbrace{x_0^2}+\underbrace{2x_0x_1\varepsilon}+\underbrace{(2x_0x_2+x_1^2)\varepsilon^2}\ \underbrace{-1}\ \underbrace{-\varepsilon e^{x_0}}\ \underbrace{-\varepsilon^2 x_1 e^{x_0}} + O(\varepsilon^3) = 0$$

Matching powers of ε

(5) $\quad x_0^2 - 1 = 0 \qquad O(1)$

(6) $\quad 2x_0x_1 - e^{x_0} = 0 \qquad O(\varepsilon)$

(7) $\quad 2x_0x_2 + x_1^2 - e^{x_0}x_1 = 0 \qquad O(\varepsilon^2)$

For smaller root $\bar{x}_-$ we have $x_0 = -1$ which when used in (6)-(7) yields

$$\bar{x}_-(\varepsilon) = -1 - \tfrac{1}{2}e^{-1}\varepsilon + \tfrac{3}{8}e^{-2}\varepsilon^2 + \cdots$$

Similarly for $\bar{x}_+$ we have $\bar{x}_+(0) = x_0 = +1$

$$\bar{x}_+(\varepsilon) = +1 + \tfrac{1}{2}e\,\varepsilon + \tfrac{1}{8}e^2\varepsilon^2 + \cdots$$

These are approximate solns to a problem with no known "exact" soln (formula).

Regular Solns (Algebraic) - General Theory

Seek a regular soln $\bar{x}(\varepsilon)$ of

(1) $$f(x, \varepsilon) = 0$$

where

(2) $$\bar{x}(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$$

or equivalently by Taylor's Thm

(3) $$\bar{x}(\varepsilon) = \bar{x}(0) + \varepsilon \bar{x}'(0) + \frac{1}{2!} \bar{x}''(0) \varepsilon^2 + \cdots$$

Define

$$F(\varepsilon) \equiv f(\bar{x}(\varepsilon), \varepsilon)$$

This function has a Taylor series which must vanish

$$F(\varepsilon) = F(0) + F'(0)\varepsilon + \frac{1}{2} F''(0) \varepsilon^2 + \cdots = 0$$

Using the chain rule

(4) $$F'(\varepsilon) = f_x(\bar{x}(\varepsilon), \varepsilon)\bar{x}'(\varepsilon) + f_\varepsilon(\bar{x}(\varepsilon), \varepsilon)$$

Thus, since $\bar{x}(0) = x_0$, $\bar{x}'(0) = x_1$ we have

(5) $$F(0) = f(x_0, 0) = 0 \qquad O(1)$$

(6) $$F'(0) = f_x(x_0, 0)\, x_1 + f_\varepsilon(x_0, 0) = 0 \qquad O(\varepsilon)$$

are the first two of a sequence of eqns for finding $x_0, x_1, x_2 \ldots$ from setting all derivatives of F equal to zero.

To $O(\varepsilon^2)$ one must compute $F''(\varepsilon)$

$$F''(\varepsilon) = f_{xx}(\bar{x}')^2 + 2f_{x\varepsilon}\bar{x}' + f_x \bar{x}'' + f_{\varepsilon\varepsilon}$$

Noting $x_2 = \frac{1}{2}\bar{x}''(0)$ by comparing (2)-(3) we find

$$(7)\quad F''(0) = f_{xx}^{(0)} x_1^2 + 2f_{x\varepsilon}^{(0)} x_1 + 2f_x^{(0)} x_2 + f_{\varepsilon\varepsilon}^{(0)}$$

where $(\ )^{(0)}$ denotes evaluation at $(x_0, 0)$.

Having found x_0, x_1, equation (7) can be used to find x_2 from the condition $F''(0) = 0$.

EXAMPLE $\boxed{\text{Larger root of } f(x,\varepsilon) = x^2 - 1 - \varepsilon(3+x)^{-1/2}}$

$$f_x(x,\varepsilon) = 2x + \frac{1}{2}(3+x)^{-3/2}\varepsilon \qquad f_x(x_0, 0) = 2x_0$$

$$f_\varepsilon(x,\varepsilon) = -(3+x)^{-1/2} \qquad f_\varepsilon(x_0, 0) = -(3+x_0)^{-1/2}$$

The leading equation is

$$f(x_0, 0) = x_0^2 - 1 = 0 \qquad \Rightarrow x_0 = 1$$

Then, general theory and $f_x(1,0) = 2$, $f_\varepsilon(1,0) = -\frac{1}{2}$

$$f_x(x_0,0)\, x_1 + f_\varepsilon(x_0,0) = 0$$

$$2x_1 - \tfrac{1}{2} = 0$$

$$x_1 = \tfrac{1}{4}$$

and $\bar{x}(\varepsilon) = 1 + \frac{1}{4}\varepsilon + O(\varepsilon^2)$