Regular Perturbations (Algebraic)

Let \( f : \mathbb{R} \times I \to \mathbb{R}, \varepsilon \in I \). Seek a soln in \( x \) of

(1) \[ f(x, \varepsilon) = 0 \quad x \in \mathbb{R} \]

of the form

(2) \[ x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots \]

The expansion (2) is said to be regular in \( \varepsilon \).

Eqn (2) is an assumption that the solution \( x(\varepsilon) \) is smooth in \( \varepsilon \) and as a consequence has a Taylor series expansion, i.e.,

\[ x(\varepsilon) = x(0) + \varepsilon x'(0) + \frac{1}{2} \varepsilon^2 x''(0) + \cdots \]

whether (2) is a valid assumption depends on \( f \).

One theorem addressing this: is the Implicit Function Thm.

**Theorem**

Let \( f : \mathbb{R} \times I \to \mathbb{R} \) and assume \( f \) and all its partial derivatives are smooth. Then if

(a) \[ f(\bar{x}, 0) = 0 \]

(b) \[ f_x(\bar{x}, 0) \neq 0 \]

then \( \exists \) a smooth fn \( x(\varepsilon) \) with \( x(0) = \bar{x} \) such that \( f(x(\varepsilon), \varepsilon) = 0 \) for all \( \varepsilon \) in some neighborhood of \( \varepsilon = 0 \).
EXAMPLE \[ X^3 + \varepsilon X - 1 = 0, \quad 0 < \varepsilon \ll 1 \]

Assume

(1) \[ X = X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \cdots \]

Use (1) in \( \frac{f(x, \varepsilon)}{\varepsilon} = 0 \)

\[ (X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + \cdots)^3 + \varepsilon (X_0 + \varepsilon X_1 + \cdots) - 1 = 0 \]

Expand and collect in powers of \( \varepsilon \).

\[ (X_0^3 + 3X_0^2X_1\varepsilon + (3X_0X_1^2 + 3X_0^2X_2)\varepsilon^2) + \varepsilon (X_0 + \varepsilon X_1 + \cdots) - 1 = 0 \]

Arrive at a set of eqns which can be solved.

\( O(1) \) \[ X_0^3 - 1 = 0 \] solve for \( X_0 \)

\( O(\varepsilon) \) \[ 3X_0^2X_1 + X_0 = 0 \] then \( X_1 \)

\( O(\varepsilon^2) \) \[ 3X_0^2X_2 + 3X_0X_1^2 + X_1 = 0 \] then \( X_2 \)

Solving sequentially

\[ X_0 = 1 \]

\[ X_1 = -\frac{1}{3} \]

\[ X_2 = 0 \]

To conclude

\[ x(\varepsilon) = 1 - \frac{1}{3} \varepsilon + O(\varepsilon^3) \]
EXAMPLE

\[ x^2 - 1 - \varepsilon e^x = 0, \quad 0 < \varepsilon \ll 1 \]

\[ x^2 - 1 = \varepsilon e^x \]

\text{intersection of curves}

\text{clearly two roots}

Seek a regular approximation for both roots.

(1) \[ \bar{x}(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3) \]

Need an expansion for \( e^{\bar{x}(\varepsilon)} \). First note

(2) \[ e^{x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots} = e^{x_0} e^{\varepsilon (x_1 + \varepsilon x_2 + \cdots)} \]

Letting \( z = \varepsilon (x_1 + \varepsilon x_2 + \cdots) \) which is small

\[ e^z = 1 + z + \frac{1}{2} z^2 + \cdots \]

\[ e^z = 1 + (\varepsilon x_1 + \varepsilon^2 x_2 + \cdots) + \frac{1}{2} (\varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^2 \]

(3) \[ e^z = 1 + \varepsilon x_1 + (x_2 + \frac{1}{2} x_1^2) \varepsilon^2 + O(\varepsilon^3) \]

Using (3) in (2) we find

(4) \[ e^{\bar{x}(\varepsilon)} = e^{x_0} + \varepsilon x_1 e^{x_0} + (x_2 + \frac{1}{2} x_1^2) e^{x_0} \varepsilon^2 + \cdots \]

Now use (4) and (1) in \( f(x, \varepsilon) = 0 \), expand and collect powers of \( \varepsilon \).
\[(x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots)^2 \quad -1 - \varepsilon (e^{x_0} + \varepsilon x_1 e^{x_0} + O(\varepsilon^2)) = 0\]

\[x_0^2 + \frac{2x_2}{x_1} + 2x_0 x_1 \varepsilon + (2x_0 x_2 + x_1^2) \varepsilon^2 \quad -1 - \frac{\varepsilon e^{x_0} - \varepsilon^2 x_1 e^{x_0} + O(\varepsilon^3)}{} = 0\]

Matching powers of \(\varepsilon\)

(5) \[x_0^2 - 1 = 0 \quad O(1)\]

(6) \[2x_0 x_1 - e^{x_0} = 0 \quad O(\varepsilon)\]

(7) \[2x_0 x_2 + x_1^2 - e^{x_0} x_1 = 0 \quad O(\varepsilon^2)\]

For smaller root \(\bar{x}_-\) we have \(x_0 = -1\)
which when used in (6) - (7) yields
\[\bar{x}_-(\varepsilon) = -1 - \frac{1}{2} e^{-1} \varepsilon + \frac{3}{8} e^{-2} \varepsilon^2 + \cdots\]

Similarly for \(\bar{x}_+\) we have \(\bar{x}_+(0) = x_0 = +1\)
\[\bar{x}_+(\varepsilon) = +1 + \frac{1}{2} e \varepsilon + \frac{1}{8} e^2 \varepsilon^2 + \cdots\]

These are approximate solns to a problem with no known "exact" soln (formula).
Regular Solns (Algebraic) - General Theory

Seek a regular soln \( \bar{x}(\varepsilon) \) of

\[ f(x, \varepsilon) = 0 \]

where

\[ \bar{x}(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots \]

or equivalently by Taylor's Thm

\[ \bar{x}(\varepsilon) = \bar{x}(0) + \varepsilon \bar{x}'(0) + \frac{1}{2!} \bar{x}''(0) \varepsilon^2 + \cdots \]

Define

\[ F(\varepsilon) = f(\bar{x}(\varepsilon), \varepsilon) \]

This function has a Taylor series which must vanish

\[ F(\varepsilon) = F(0) + F'(0) \varepsilon + \frac{1}{2} F''(0) \varepsilon^2 + \cdots = 0 \]

Using the chain rule

\[ F'(\varepsilon) = f_x(\bar{x}(\varepsilon), \varepsilon) \bar{x}'(\varepsilon) + f_\varepsilon(\bar{x}(\varepsilon), \varepsilon) \]

Thus, since \( \bar{x}(0) = x_0, \bar{x}'(0) = x_1 \), we have

\[ F(0) = f(x_0, 0) = 0 \quad O(1) \]

\[ F'(0) = f_x(x_0, 0) x_1 + f_\varepsilon(x_0, 0) = 0 \quad O(\varepsilon) \]

are the first two of a sequence of eqns for finding \( x_0, x_1, x_2 \ldots \) from setting all derivatives of \( F \) equal to zero.
To $O(\varepsilon^2)$ one must compute $F''(\varepsilon)$

$$F''(\varepsilon) = \int_{x} (\bar{x})^2 + 2 \int_{x \varepsilon} \bar{x} + \int_{\varepsilon} \bar{x}^2 + \int_{\varepsilon \varepsilon}$$

Noting $x_2 = \frac{1}{2} \bar{x}''(0)$ by comparing (2) - (3) we find

(7) $F''(0) = \int_{x} (0) x_1^2 + 2 \int_{x \varepsilon} (0) x_1 + 2 \int_{\varepsilon} (0) x_2 + \int_{\varepsilon \varepsilon}$

where ( )$^{(0)}$ denotes evaluation at $(x_0, 0)$.

Having found $x_0, x_1$ equation (7) can be used to find $x_2$ from the condition $F''(0) = 0$.

**Example**

Larger root of $f(x, \varepsilon) = x^2 - 1 - \varepsilon (3 + x)^{-\frac{1}{2}}$

\[
\begin{align*}
  f_x(x, \varepsilon) &= 2x + \frac{1}{2} (3 + x)^{-\frac{3}{2}} \varepsilon \\
  f_x(x_0, 0) &= 2x_0 \\
  f_\varepsilon(x, \varepsilon) &= -(3 + x)^{-\frac{1}{2}} \\
  f_\varepsilon(x_0, 0) &= -(3 + x_0)^{-\frac{1}{2}}
\end{align*}
\]

The leading equation is

$$f(x_0, 0) = x_0^2 - 1 = 0 \quad \Rightarrow \quad x_0 = 1$$

Then, general theory and $f_x(1, 0) = 2, f_\varepsilon(1, 0) = -\frac{1}{2}$

$$f_x(x_0, 0) x_1 + f_\varepsilon(x_0, 0) = 0$$

$$2x_1 - \frac{1}{2} = 0$$

$$x_1 = \frac{1}{4}$$

and $x(\varepsilon) = 1 + \frac{1}{4} \varepsilon + O(\varepsilon^2)$