Regular Perturbation Theory (Differential Eqns)

Seek solution expansions of the form

(1) $y(t,\varepsilon) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots$

to problems like

(2) $F(y'', y', y, t, \varepsilon) = 0 \qquad 0 < \varepsilon \ll 1$

May also include initial conditions

(3) $y(0,\varepsilon) = a \qquad y'(0,\varepsilon) = b$

where a, b could also depend on ε.

For instance

$$y'' + \varepsilon \sin t \, y' + y = y^2$$
$$y(0) = 1 \qquad y'(0) = 2 + \varepsilon$$

The method of solution involves substituting the expansion (1) into the differential equation and initial conditions, expanding and then collecting like powers of ε.

EXAMPLE $\boxed{y'' + y = \varepsilon y^2 \qquad y(0) = 1 \qquad y'(0) = 0}$

Assume

$$y(t, \varepsilon) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots$$

Expand initial conditions

$$y(0, \varepsilon) = y_0(0) + \varepsilon y_1(0) + \cdots = 1$$

$$y'(0, \varepsilon) = y_0'(0) + \varepsilon y_1'(0) + \cdots = 0$$

From which $y_K'(0) = 0$, $y_0(0) = 1$, $y_K(0) = 0$ for $K \geq 1$.

Expand differential equation

$$(y_0'' + \varepsilon y_1'' + \cdots) + (y_0 + \varepsilon y_1 + \cdots) \pm \varepsilon (y_0 + \varepsilon y_1 + \cdots)^2$$

Collecting powers of ε and using initial conditions we arrive at a sequence of initial value problems

(1) $\quad y_0'' + y_0 = 0 \qquad y_0(0) = 1 \qquad y_0'(0) = 0$

(2) $\quad y_1'' + y_1 = y_0^2 \qquad y_1(0) = 0 \qquad y_1'(0) = 0$

Clearly $y_0(t) = \cos t$. Then (2) becomes

$$y_1'' + y_1 = \cos^2 t = \frac{1}{2} + \frac{1}{2}\cos t$$

Using undetermined coefficients (method)

$$y_1(t) = \frac{1}{2} - \frac{1}{3}\cos t - \frac{1}{6}\cos 2t$$

So that

$$y(t, \varepsilon) = \cos t + \varepsilon\left(\frac{1}{2} - \frac{1}{3}\cos t - \frac{1}{6}\cos 2t\right) + O(\varepsilon^2)$$

EXAMPLE $\boxed{y'y'' - \varepsilon y^2 = 0, \quad y(0) = 0 \quad y'(0) = 1}$

Assume

$$y(t,\varepsilon) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots$$

Initial condition expansions are clear

$$y_0(0) = 0 \qquad y_0'(0) = 1$$

$$y_1(0) = 0 \qquad y_1'(0) = 0$$

Expand differential equation

$$(y_0' + \varepsilon y_1' + \cdot)(y_0'' + \varepsilon y_1'' + \cdot) - \varepsilon(y_0 + \varepsilon y_1 + \cdot)^2 = 0$$

Collecting like powers

$$O(1) \qquad y_0' y_0'' = 0$$

$$O(\varepsilon) \qquad y_0' y_1'' + y_0'' y_1 = y_0^2$$

are nonlinear equations

Solving $O(1)$ problem

$$y_0' y_0'' = 0 \qquad y_0(0) = 0 \quad y_0'(0) = 1$$

Note that

$$\frac{d}{dt}(y_0'^2) = 0$$

Thus y_0' is constant and

$$y_0(t) = At + B$$

Applying initial conditions

$$y_0'(0) = A = 1$$
$$y_0(0) = B = 0$$

Conclude

$$\boxed{y_0(t) = t}$$

<u>Solving $O(\varepsilon)$ problem</u>

Using $y_0(t) = t$ in $O(\varepsilon)$ eqns we arrive at

$$y_1'' = t^2 \qquad y(0) = 0 \qquad y_1'(0) = 0$$

whose general solution is

$$y_1(t) = \frac{1}{12} t^4 + At + B$$

for constants A, B.

$$y_1(0) = B = 0$$
$$y_1'(0) = A = 0$$

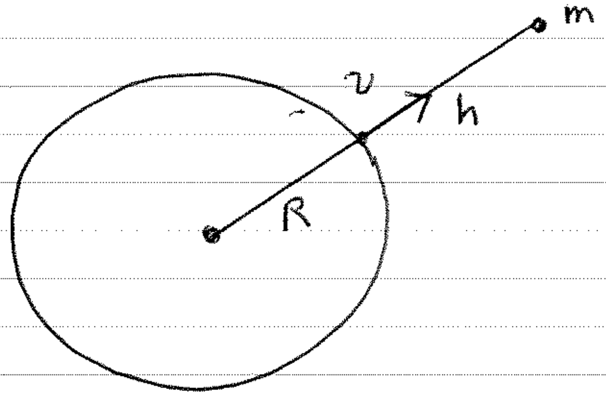
Conclude

$$y_1(t) = \frac{1}{12} t^4$$

and

$$y(t, \varepsilon) = t + \tfrac{1}{2} \varepsilon t^4 + O(\varepsilon^2)$$

Motivational Example - Projectile Problem



A projectile of mass m is launched from earth surface at a velocity v.

Earth has radius R and mass M. G is the universal gravitational constant.

The height $h(T)$ of the projectile is found using Newtons law of gravity

$$(1) \qquad m\frac{d^2h}{dT^2} = -\frac{GMm}{(h+R)^2}, \quad h(0)=0, \quad h'(0)=v$$

Convert this dimensional IVP into a dimensionless one

$$y = \frac{h}{H^*} \qquad t = \frac{T}{T^*}$$

yields

$$\frac{d^2y}{dt^2} = -\left(\frac{GMT^{*2}}{H^*R^2}\right)\frac{1}{\left(1+\frac{H^*}{R}y\right)^2} \qquad y'(0) = \frac{vT^*}{H^*}$$

choose H^* and T^* so these both equal 1

yields

$$\boxed{T^* = \frac{vR^2}{GM} = \frac{v}{g} \qquad H^* = \frac{v^2R^2}{GM}}$$

Dimensionless problem -(what's small?)

$$\boxed{(2)\quad y'' = -\frac{1}{(1+\varepsilon y)^2} \qquad y(0)=0\,,\; y'(0)=1}$$

where the sole parameter is

$$(3) \qquad \varepsilon = \frac{v^2 R}{GM} \ll 1$$

To see why ε might be small note that the gravitational constant g at the surface is

$$g = \frac{GM}{R^2} = 9.8\ m/sec^2$$

Then ε can be written

$$(4) \qquad \varepsilon = \frac{v^2}{gR} \ll 1$$

owing to the fact that the radius R of the earth is large

$$R \simeq 6.4 \times 10^6 m \qquad g = 9.8\ m/sec^2$$

A couple of examples

$\varepsilon \simeq 0.00016$ $\qquad v = 100\ m/sec$

$\varepsilon \simeq 0.017$ $\qquad v = 1000\ m/sec$ (Mach 3.0)

Regular series approximation

(1) $$y'' = -\frac{1}{(1+\varepsilon y)^2}, \quad 0 < \varepsilon \ll 1$$

(2) $$y(0) = 0$$

(3) $$y'(0) = 1$$

Assume

$$y(t,\varepsilon) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + O(\varepsilon^3)$$

Expand initial conditions

$$y(0,\varepsilon) = y_0(0) + \varepsilon y_1(0) + \cdots = 0$$

$$y'(0,\varepsilon) = y_0'(0) + \varepsilon y_1'(0) + \cdots = 1$$

yields

$$y_0(0) = 0 \qquad y_0'(0) = 1$$

$$y_1(0) = 0 \qquad y_1'(0) = 0$$

To expand the differential equation we use the Binomial Theorem

$$\frac{1}{(1+\varepsilon y)^2} = (1+\varepsilon y)^{-2} = 1 - 2\varepsilon y + 3\varepsilon^2 y^2 + \cdots$$

Recall Binomial Thm

$$(1+z)^p = 1 + pz + \frac{1}{2!}p(p-1)z^2 + \cdots$$

if $|z| < 1$

$$\frac{1}{(1+\varepsilon y)^2} = 1 - 2\varepsilon(y_0 + \varepsilon y_1 + \cdot\cdot) + 3\varepsilon^2(y_0 + \varepsilon y_1 + \cdot\cdot)^2 + \cdots$$

$$= 1 - 2\varepsilon y_0 + (3y_0^2 - 2y_1)\varepsilon^2 + O(\varepsilon^2)$$

Since

$$y'' = y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \cdots$$

the expansions above yield (by equating like powers of ε) the following series of initial value problems

$$(4) \quad y_0'' = -1 \qquad y_0(0) = 0 \qquad y_0'(0) = 1$$

$$(5) \quad y_1'' = 2y_0 \qquad y_1(0) = 0 \qquad y_1'(0) = 0$$

$$(6) \quad y_2'' = -3y_0^2 + 2y_1 \qquad y_2(0) = 0 \qquad y_2'(0) = 0$$

whose solutions are

$$y_0(t) = -\tfrac{1}{2}t^2 + t$$

$$y_1(t) = -\frac{1}{12}t^3(t-4)$$

$$y_2(t) = -\frac{1}{360}t^4(11t^2 - 66t + 90)$$

Summary and application

$$y(t,\varepsilon) = \left(t - \tfrac{1}{2}t^2\right) - \tfrac{1}{12}\varepsilon t^3(t-4) + O(\varepsilon^2)$$

can convert back to dimensional form using

$$y = \frac{g}{v^2} h \qquad t = \frac{g}{v} T$$

Maximal height correction

Seek an approximation to the time the projectile takes to achieve its maximal height. This occurs when its velocity is zero:

$$y' = f(t,\varepsilon) = 1 - t + \varepsilon\left(t^2 - \tfrac{1}{3}t^3\right) + \cdots = 0$$

Let

$$t = \tau(\varepsilon) = \tau_0 + \varepsilon\tau_1 + O(\varepsilon^2)$$

Substitution and collection of powers yields

$$(1 - \tau_0) + \varepsilon\left(\tau_0^2 - \tfrac{1}{3}\tau_0^3 - \tau_1\right) + O(\varepsilon^2) = 0$$

From which $\tau_0 = 1$, $\tau_1 = \frac{2}{3}$ and

$$\tau = 1 + \tfrac{2}{3}\varepsilon + O(\varepsilon^2)$$

represents a slightly increased time to maximum height — as one would expect since gravity weakens the higher one goes.