Regular Perturbations - Nonlinear Oscillations

Depending on initial conditions and the function f, the solution $y(t,\varepsilon)$ of

$$y'' + y = \varepsilon f(y, y', t) \quad , \; 0 < \varepsilon \ll 1$$

may or may not be periodic.

When the exact solution $y(t,\varepsilon)$ is periodic, regular perturbation techniques yield undesired results!

We illustrate this issue by way of example

EXAMPLE $\quad y'' + y = \varepsilon y \qquad y(0) = 0 \quad y'(0) = 1$

The exact solution is

$$y(t,\varepsilon) = \frac{\sin(\sqrt{1-\varepsilon}\, t)}{\sqrt{1-\varepsilon}}$$

Now suppose we assumed

$$y(t,\varepsilon) = y_0(t) + \varepsilon y_1(t) + \cdots$$

Then

$O(1) \qquad y_0'' + y_0 = 0 \qquad y_0(0) = 0 \quad y_0'(0) = 1$

$O(\varepsilon) \qquad y_1'' + y_1 = y_0 \qquad y_1(0) = 0 \quad y_1'(0) = 0$

The solution of the $O(1)$ problem is

$$y_0(t) = \sin t$$

The simplified $O(\varepsilon)$ problem is

$$y_1'' + y_1 = \sin t \qquad y_1(0) = y_1'(0) = 0$$

Since $\sin t$ on the right side is part of the homogeneous soln one should seek particular solns of the form

$$y_1^p(t) = At\sin t + Bt\cos t$$

One finds $A=0$, $B=-\frac{1}{2}$ via undetermined coefficients.

Ultimately

$$y_1(t) = c_1 \cos t + c_2 \sin t - \frac{1}{2} t\cos t$$

and the initial conditions lead to

$$y_1(t) = \frac{1}{2}\sin t - \frac{1}{2} t\cos t$$

Summary of regular series approximation:

$$y(t,\varepsilon) = \sin t + \varepsilon\left(\frac{1}{2}\sin t - \underbrace{\frac{1}{2}t\cos t}_{\text{Term grows in time even though exact soln does not.}}\right) + O(\varepsilon^2)$$

Although the expansion is not periodic, the exact soln is!

However the expansion can be verified using the exact soln

$$y(t,\varepsilon) = (1-\varepsilon)^{-1/2}\sin((1-\varepsilon)^{1/2}t) \qquad \omega \equiv \sqrt{1-\varepsilon}$$

$$y_\varepsilon(t,\varepsilon) = \tfrac{1}{2}(1-\varepsilon)^{-3/2}\sin(\omega t) - \tfrac{1}{2}(1-\varepsilon)^{-1}\cos(\omega t)$$

From which

$$y(t,\varepsilon) = y(t,0) + y_\varepsilon(t,0)\,\varepsilon + O(\varepsilon^2)$$

$$(1) \quad y(t,\varepsilon) = \sin t + \varepsilon\left(\tfrac{1}{2}\sin t - \tfrac{1}{2}t\cos t\right) + O(\varepsilon^2).$$

The series (1) _is_ an approximation valid for $t=O(1)$, i.e t values not too large.

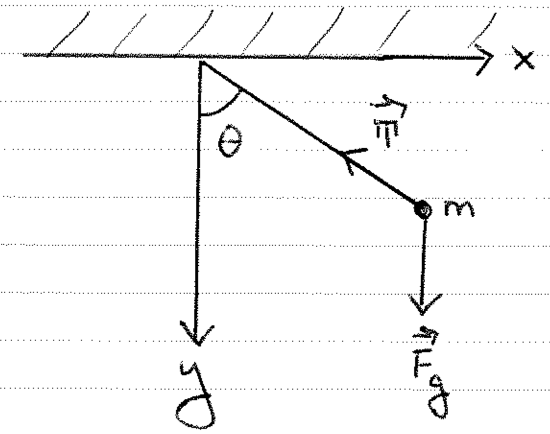
The series (1) fails to be a good approximation when $t = O(\frac{1}{\varepsilon})$. Then the term

$$\varepsilon t \cos t \qquad \text{"secular term"}$$

is $O(1)$. Such terms that grow and change order are said to be _secular_.

Pendulum Motion Derivation

Mass m swings on a wire of length L



Here

$\vec{F}_g$ = gravitational force

$\vec{T}$ = tension in wire

For the chosen coordinate orientation

$$\vec{F}_g = \langle 0, +mg \rangle$$

$$\vec{T} = \langle -T\sin\theta, -T\cos\theta \rangle$$

and $T \equiv |\vec{T}|$.

Also,

(1) $\quad x = L\sin\theta \qquad y = L\cos\theta$

which we use to compute the acceleration

$$\vec{a} = \langle \ddot{x}, \ddot{y} \rangle \qquad \dot{(\ )} = \frac{d}{dt}(\ )$$

Differentiation of (1) in t twice

(2) $\ddot{x} = -L\sin\theta\,\dot{\theta}^2 + L\cos\theta\,\ddot{\theta}$

(3) $\ddot{y} = -L\cos\theta\,\dot{\theta}^2 - L\sin\theta\,\ddot{\theta}$

Newtons Laws of Motion require

$$m\vec{a} = \vec{F}_g + \vec{T}$$

Written in component form

(4) $-mL\sin\theta\,\dot{\theta}^2 + mL\cos\theta\,\ddot{\theta} = -T\sin\theta$

(5) $-mL\cos\theta\,\dot{\theta}^2 - mL\sin\theta\,\ddot{\theta} = mg - T\cos\theta$

Then the algebra $\cos\theta$ Eqn (4) $-\sin\theta$ Eqn(5) yields

$$\boxed{\ddot{\theta} + \frac{g}{L}\sin\theta = 0} \qquad (\dot{\ }) = \frac{d}{d\tau}(\ )$$

Let dimensionless time

$$t = \tau/\tau^* \qquad \tau^* = \sqrt{\frac{L}{g}}$$

yields the dimensionless equation

(6) $$\boxed{\frac{d^2\theta}{dt^2} + \sin\theta = 0}$$

Small amplitude oscillations - Duffing's Eqn

Seek small amplitude (angle) approximations to the pendulum problem

$$(1) \qquad \theta'' + \sin\theta = 0$$

Let $0 < \delta \ll 1$ and

$$\theta(t) = \delta\, u(t)$$

Then using

$$\sin(\delta u) = \delta u - \frac{1}{3!}\delta^3 u^3 + O(\delta^5)$$

in equation (1) we have

$$\cancel{\delta} u'' + \cancel{\delta} u - \frac{1}{3!}\delta^{\cancel{3}} u^3 + O(\delta^{\cancel{5}}) = 0$$

or

$$(2) \qquad u'' + u = \frac{1}{6}\varepsilon u^3 + O(\varepsilon^2)$$

where $\varepsilon = \delta^2 \ll 1$. Setting $u = \sqrt{6}\, y$ we arrive at (dropping $O(\varepsilon^2)$ terms)

$$(3) \qquad \boxed{y'' + y = \varepsilon y^3}$$

Duffing's Equation

Must be solved to obtain next order correcting to pendulum problem.

Assume a regular series approximation

$$y(t,\varepsilon) = y_0(t) + \varepsilon y_1(t) + O(\varepsilon^2)$$

yields

(4) $$y_0'' + y_0 = 0 \qquad y_0(0) = 1,\ y_0'(0) = 0$$

(5) $$y_1'' + y_1 = y_0^3 \qquad y_1(0) = 0,\ y_1'(0) = 0$$

Soln of $O(1)$ problem is clearly

$$y_0(t) = \cos t$$

To solve (5) one makes use of the method of undetermined coefficients to find a particular soln. That and the identity

$$\cos^3 t = \frac{3}{4}\cos t + \frac{1}{4}\cos 3t$$

After some work,

$$y_1(t) = \frac{1}{32}(\cos t - \cos 3t) + \frac{3}{8} t \sin t$$

To conclude

$$y(t,\varepsilon) = \cos t + \frac{1}{8}\varepsilon\left\{ \frac{1}{4}(\cos t - \cos 3t) + 3t\sin t \right\} + \cdots$$

Secular Term grows in time even though soln doesn't

Poincaré - Lindstedt Method

Seek a periodic approximation of

(1) $y'' + y = \varepsilon f(y, y')$

(2) $y(0) = a$

(3) $y'(0) = b$

by assuming

$$y(t, \varepsilon) = y_0(\tau) + \varepsilon y_1(\tau) + \varepsilon^2 y_2(\tau) + \cdots$$

for a "strained" or "scaled" time

$$\tau \equiv \omega(\varepsilon) t$$

where

$$\omega(\varepsilon) = 1 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \cdots$$

Object: Choose ω_K so $y_K(\tau)$ are periodic in τ.

The resulting approximation will be

$$T = \frac{2\pi}{\omega(\varepsilon)}$$

periodic in the original (nonstrained) time t.

If the original diff eqn is $y'' + \omega_0^2 y = \varepsilon f(y, y')$ the one can (instead) use

$$\omega(\varepsilon) = \omega_0 + \varepsilon \omega_1 + \varepsilon^2 \omega_2 + \cdots$$

Transforming Derivatives

Define the strained time τ by

$$\tau = \omega(\varepsilon)t$$

and re-express $y(t,\varepsilon)$ in terms of (τ,ε).

$$Y(\tau,\varepsilon) = y(t,\varepsilon)$$

Using the chain rule

$$\frac{dy}{dt} = \omega\frac{dY}{d\tau}$$

$$\frac{d^2y}{dt^2} = \omega^2\frac{d^2Y}{d\tau^2}$$

Now let

$$y(t,\varepsilon) = Y(\tau,\varepsilon) = y_0(\tau) + \varepsilon y_1(\tau) + \varepsilon^2 y_2(\tau) + \cdots$$

Then the time derivatives of y are found by expanding the following

$$\frac{dy}{dt} = (1+\varepsilon\omega_1+\varepsilon^2\omega_2+\cdots)(y_0' + \varepsilon y_1' + \varepsilon^2 y_2' + \cdots)$$

$$\frac{d^2y}{dt^2} = (1+\varepsilon\omega_1+\varepsilon^2\omega_2+\cdots)^2(y_0'' + \varepsilon y_1'' + \varepsilon^2 y_2'' + \cdots)$$

Calculations reveal (to $O(\varepsilon^3)$)

$$\frac{dy}{dt} = y_0' + (y_1' + \omega_1 y_0')\varepsilon + (y_2' + \omega_1 y_1' + \omega_2 y_0')\varepsilon^2 + \cdots$$

$$\frac{d^2y}{dt^2} = y_0'' + (y_1'' + 2\omega_1 y_0'')\varepsilon + \Lambda_2 \varepsilon^2 + \cdots$$

where the $O(\varepsilon^2)$ coefficient for y'' is

$$\Lambda_2 = y_2'' + 2\omega_1 y_1'' + (2\omega_2 + \omega_1^2) y_0''$$

Transformation of initial conditions

Since $y(0,\varepsilon) = a$ we have

$$y(t,\varepsilon) = y_0(\tau) + \varepsilon y_1(\tau) + \varepsilon^2 y_2(\tau) + \cdots$$

$$a = y_0(0) + \varepsilon y_1(0) + \varepsilon^2 y_2(0) + \cdots$$

From which we conclude

$$y_0(0) = a$$

$$y_K(0) = 0 \qquad K = 1, 2, 3 \ldots$$

Initial conditions involving derivatives are different.

Recall

$$y'(t,\varepsilon) = y_0'(\tau) + (y_1'(\tau) + \omega_1 y_0'(\tau))\varepsilon + O(\varepsilon^2)$$
$$y'(0,\varepsilon) = y_0'(0) + (y_1'(0) + \omega_1 y_0'(0))\varepsilon + O(\varepsilon^2)$$
$$b = y_0'(0) + (y_1'(0) + \omega_1 y_0'(0))\varepsilon + O(\varepsilon^2)$$

Equating powers of ε

$$y_0'(0) = b$$
$$y_1'(0) = -\omega_1 b$$

In particular, $y_1'(0)$ will not generally be zero.

The initial condition for $y_1(\tau)$ depends on ω_1 which we don't know yet.

EXAMPLE $\boxed{y'' + y = \varepsilon y \ , \ y(0) = 0 \ , \ y'(0) = 1}$

Let

$$y(t,\varepsilon) = y_0(\tau) + \varepsilon y_1(\tau) + \cdots$$
$$\tau = (1 + \omega_1 \varepsilon + \cdots) t$$

Expand differential equation

$$(1+\omega_1\varepsilon + \cdots)^2 (y_0'' + \varepsilon y_1'' + \cdot) + (y_0 + \varepsilon y_1 + \cdot) = \varepsilon (y_0 + \varepsilon y_1 + \cdot)$$

$$(y_0'' + y_0) + (y_1'' + y_1 + 2\omega_1 y_0'')\varepsilon = \varepsilon y_0 + O(\varepsilon^2)$$

yields an $O(1)$ and $O(\varepsilon)$ eqn.

$$y_0'' + y_0 = 0$$
$$y_1'' + y_1 = y_0 - 2\omega_1 y_0''$$

Expand $y(0) = 0$

$$y(0,\varepsilon) = 0 = y_0(0) + \varepsilon y_1(0) + O(\varepsilon^2)$$

from which we conclude

$$y_0(0) = 0$$
$$y_1(0) = 0$$

Expand $y'(0) = 1$

$$y'(t) = (1 + \varepsilon\omega_1 + \cdots)(y_0'(\tau) + \varepsilon y_1'(\tau) + \cdots)$$

$$1 = y_0'(0) + (y_1'(0) + \omega_1 y_0'(0))\,\varepsilon + O(\varepsilon^2)$$

from which we conclude

$$y_0'(0) = 1$$

$$y'(0) = -\omega_1$$

Summary of $O(1)$ and $O(\varepsilon)$ problems

$O(1)$ $\quad y_0'' + y_0 = 0 \quad , y_0(0) = 0,\ y_0'(0) = 1$

$O(\varepsilon)$ $\quad y_1'' + y_1 = y_0 - 2\omega_1 y_0'' \ , y_1(0) = 0,\ y_1'(0) = -\omega$

Soln of $O(1)$ problem is

$$y_0(\tau) = \sin\tau$$

when used in $O(\varepsilon)$ problem

$$y_1'' + y_1 = \underbrace{(1 + 2\omega_1)}\sin\tau$$

CRITICAL POINT HERE.
IF WE CHOOSE $(1 + 2\omega_1) = 0$
THEN y_1 WILL HAVE NO SECULAR OR GROWING TERMS.

Choose

$$\omega_1 = -\tfrac{1}{2}$$

So $y_1(\tau)$ is periodic in τ, hence bounded and not secular (growing)

Then the $O(1)$ problem is

$$y_1'' + y_1 = 0 \qquad y_1(0) = 0 \quad y_1'(0) = \tfrac{1}{2}$$

whose soln is

$$y_1(\tau) = \tfrac{1}{2} \sin \tau$$

Summary of Soln approximation

$$y = \sin \tau + \tfrac{1}{2} \varepsilon \sin \tau + O(\varepsilon^2)$$

unlike regular series approx this term doesn't grow with time. So approximation remains better, longer

where

$$\tau = (1 - \tfrac{1}{2}\varepsilon + O(\varepsilon^2))$$

Note how this compares with the known exact solution

$$y(t, \varepsilon) = \frac{\sin(\sqrt{1-\varepsilon}\, t)}{\sqrt{1-\varepsilon}}$$

Clearly the method is reproducing $\omega(\varepsilon)$ needed to modify period

$$\omega(\varepsilon) = \sqrt{1-\varepsilon} = 1 - \tfrac{1}{2}\varepsilon + O(\varepsilon^2)$$

<u>Summary of solns and Approximations</u>

The simple problem h

$$y'' + y = \varepsilon \qquad y(0) = 0 \qquad y'(0) = 1$$

has a known exact solution

<u>EXACT</u> (1) $$y(t, \varepsilon) = \frac{\sin(\sqrt{1-\varepsilon}\, t)}{\sqrt{1-\varepsilon}}$$

If one uses a regular expansion in t

$$y(t,\varepsilon) = y_0(t) + \varepsilon y_1(t) + \cdots$$

we find

<u>REGULA</u> (2) $$y(t,\varepsilon) = \sin t + \tfrac{1}{2}\varepsilon(\sin t - \underbrace{t\cos t}_{\text{grows}}) + \cdots$$

However, if one uses Poincaré-Lindstedt

$$y(t,\varepsilon) = y_0(\tau) + \varepsilon y_1(\tau) + \cdots$$
$$\tau = (1 + \omega_1 \varepsilon + \cdots)t$$

we find

<u>LINDSTED</u> (3) $$y(t,\varepsilon) = \sin\tau + \tfrac{1}{2}\varepsilon \sin\tau + \cdots$$
$$\tau = \omega(\varepsilon)t = (1 - \tfrac{1}{2}\varepsilon + \cdots)t$$

with a corrected period

$$T = \frac{2\pi}{\omega(\varepsilon)} = \frac{2\pi}{(1 - \frac{1}{2}\varepsilon + \cdots)} = 2\pi(1 + \tfrac{1}{2}\varepsilon + \cdots)$$

EXAMPLE Use Poincaré-Lindstedt's method to approximate a periodic solution of Duffings equation

$$y'' + y = \varepsilon y^3 \qquad y(0) = 1 \quad y'(0) = 0$$

Let

$$y(t,\varepsilon) = y_0(\tau) + \varepsilon y_1(\tau) + O(\varepsilon^2)$$

$$\tau = (1 + \omega_1 \varepsilon + \cdots)\, t$$

Expand differential equations

$$(1+\omega_1\varepsilon+\cdots)^2(y_0'' + \varepsilon y_1'' + \cdots) + (y_0 + \varepsilon y_1 + \cdots) = \varepsilon(y_0 + \varepsilon y_1 + \cdots)^3$$

Expand and collect like powers of ε

$O(1) \qquad y_0'' + y_0 = 0$

$O(\varepsilon) \qquad y_1'' + y_1 = y_0^3 - 2\omega_1 y_0''$

Expand initial conditions

$$y(0,\varepsilon) = 1 = y_0(0) + \varepsilon y_1(0) + O(\varepsilon^2)$$

$$y'(0,\varepsilon) = 0 = y_0'(0) + \varepsilon(y_1'(0) + \omega_1 y_0'(0)) + O(\varepsilon^2)$$

Equating powers of ε

$$y_0(0) = 1 \qquad y_0'(0) = 0$$

$$y_1(0) = 0 \qquad y_1'(0) = 0$$

Summary of $O(1)$ and $O(\varepsilon)$ problems

$$y_0'' + y_0 = 0 \qquad y_0(0) = 1 \qquad y_0'(0) = 0$$

$$y_1'' + y_1 = y_0^3 - 2\omega_1 y_0'' \qquad y_1(0) = 0 \qquad y_1'(0) = 0$$

The solution of the $O(1)$ problem is

$$\boxed{y_0(\tau) = \cos\tau}$$

Thus, the $O(\varepsilon)$ problem becomes

$$y_1'' + y_1 = \cos^3 t + 2\omega_1 \cos t$$

Need to expand $\cos^3 t$ to reveal terms that generate secular (growing) solns

$$y_1'' + y_1 = \frac{3}{4}\cos t + \frac{1}{4}\cos 3t + 2\omega_1 \cos t$$

$$= \underbrace{\left(\frac{3}{4} + 2\omega_1\right)}_{\text{choose } \omega_1 \text{ so this vanishes}} \cos t + \frac{1}{4}\cos 3t$$

Choose

$$\boxed{\omega_1 = -\frac{3}{8}}$$

then the solution of y_1 problem is

$$\boxed{y_1(\tau) = \frac{1}{32}\left(\cos\tau - \cos(3\tau)\right)}$$

Correction to oscillation period

$$T = \frac{2\pi}{\omega(\varepsilon)} = \frac{2\pi}{\left(1 - \frac{3}{8}\varepsilon + \cdots\right)} = 2\pi\left(1 + \frac{3}{8}\varepsilon + \cdots\right)$$