Regular approximations for systems

With $0 < \varepsilon \ll 1$, an example would be

$$f_1(x_1, x_2, x_3, \varepsilon) = 0$$
$$f_2(x_1, x_2, x_3, \varepsilon) = 0$$
$$f_3(x_1, x_2, x_3, \varepsilon) = 0$$

is a system of three eqns for $x_1(\varepsilon), x_2(\varepsilon), x_3(\varepsilon)$.

Defining

$$F = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

this system can be more compactly written

(1) $$F(x, \varepsilon) = 0$$

Under certain circumstances the roots of $x(\varepsilon)$ have a power series

$$x(\varepsilon) = \bar{x}_0 + \varepsilon \bar{x}_1 + O(\varepsilon^2)$$

where $\bar{x}_k \in \mathbb{R}^n$, $n = 3$. For the planar system

$$f(x, y = 0$$
$$g(x, y, \varepsilon) = 0$$

the expansions for $x, y \in \mathbb{R}$ would be

$$x = x_0 + \varepsilon x_1 + O(\varepsilon^2)$$

$$y = y_0 + \varepsilon y_1 + O(\varepsilon^2)$$

EXAMPLE

$$(1) \qquad \boxed{\begin{aligned} xy - 27 - \varepsilon\sqrt{x+1} &= 0 \\ x^2 - y - \varepsilon x &= 0 \end{aligned}}$$

We use the expansions

$$x = x_0 + \varepsilon x_1 + O(\varepsilon^2)$$

$$y = y_0 + \varepsilon y_1 + O(\varepsilon^2)$$

in the system (1) and expand in $\varepsilon \ll 1$

$$(x_0 + \varepsilon x_1 + \ldots)(y_0 + \varepsilon y_1 + \cdots) - 27 - \varepsilon\sqrt{1 + x_0 + \varepsilon x_1 + \cdots} = 0$$

$$(x_0 + \varepsilon x_1 + \cdots)^2 - (y_0 + \varepsilon y_1 + \cdots) - \varepsilon(x_0 + \varepsilon x_1 + \cdots) = 0$$

Collecting in powers of ε yields $O(1)$ and $O(\varepsilon)$ problems

$$O(1) \qquad \boxed{\begin{aligned} x_0 y_0 - 27 &= 0 \\ x_0^2 - y_0 &= 0 \end{aligned}} \qquad \text{"leading"}$$

whose soln by inspection is $x_0 = 3$, $y_0 = 9$

$$O(\varepsilon) \qquad \boxed{\begin{aligned} y_0 x_1 + x_0 y_1 &= \sqrt{x_0 + 1} \\ 2x_0 x_1 \qquad y_1 &= x_0 \end{aligned}}$$

Using $x_0 = 3$, $y_0 = 9$ in the $O(\varepsilon)$ problem yields the <u>linear</u> problem

$$(2) \qquad 9x_1 + 3y_1 = 2$$

$$(3) \qquad 6x_1 - y_1 = 3$$

whose solution is $x_1 = \frac{11}{27}$, $y_1 = -\frac{5}{9}$ to conclude

$$x = 3 + \frac{11}{27}\varepsilon + O(\varepsilon^2)$$

$$y = 9 - \frac{5}{9}\varepsilon + O(\varepsilon^2)$$

Regular expansions of definite integrals

By way of example we seek an approximation of

$$I(\varepsilon) = \int_a^b F(x,\varepsilon)\,dx \qquad 0 < \varepsilon \ll 1$$

where $[a,b]$ closed and bounded and F is smooth.

EXAMPLE

$$I(\varepsilon) = \int_0^1 \frac{\cos(\varepsilon x)}{x^2+1}\,dx$$

For fixed x, $\cos(\varepsilon x) = 1 - \frac{1}{2!}\varepsilon^2 x^2 + \frac{1}{4!}\varepsilon^4 x^4 + O(\varepsilon^6)$

$$I(\varepsilon) = \int_0^1 \left(\frac{1}{(x^2+1)} - \frac{1}{2!}\frac{x^2}{(x^2+1)}\varepsilon^2 + \frac{1}{4!}\frac{x^4}{(x^2+1)}\varepsilon^4 + \cdots \right) dx$$

Integrate term by term to get

(1) $$I(\varepsilon) = \frac{\pi}{4} + \left(\frac{1}{2} - \frac{\pi}{8}\right)\varepsilon^2 + O(\varepsilon^4)$$

This integral can be expressed in terms of $Si(x)$ and $Ci(x)$ functions but such expressions have less value ... the $O(\varepsilon^2)$ term is very easy to understand.