Similarity Solution to Diffusion Equation

(1) $u_t = D\, u_{xx}$ $\qquad x \in \mathbb{R}^+,\ t > 0$

(2) $u(0,t) = u_0$
(3) $u(\infty,t) = 0$ $\Big\}$ Boundary Cond.
(4) $u(x,0) = 0$ $\qquad$ Initial Cond.

Here $u(x,t)$ is the concentration of a substance at position x, time t.
D is called the diffusivity.

(5) $q_1 = x \quad q_2 = t \quad q_3 = D \quad q_4 = u \quad q_5 = u_0$

Dimensionless quantities π where

$$\pi = x^{\alpha_1} t^{\alpha_2} D^{\alpha_3} u^{\alpha_4} u_0^{\alpha_5}$$

After some calculations

$$[\pi] = L^{\alpha_1 + 2\alpha_3 - 3\alpha_4 - 3\alpha_5}\, T^{\alpha_2 - \alpha_3}\, M^{\alpha_4 + \alpha_5} = 1$$

So $\vec{\alpha} \in N(A)$ where the dimension matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & -3 & -3 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{array}{ccccc} & & \uparrow & & \uparrow \end{array}$$

two free variables

so $\dim N(A) = 2$ with basis

$$\vec{\alpha}_1 = (-2, 1, 1, 0, 0)$$

$$\vec{\alpha}_2 = (0, 0, 0, -1, 1)$$

Associated dimensionless quantities

$$\Pi_1 = x^{-2} D t \qquad \Pi_2 = u^{-1} u_0$$

By the Π-theorem $\exists g$ s.t. $g(\Pi_1, \Pi_2) = 0$ or that for some F

$$\boxed{\frac{u}{u_0} = F\left(\frac{x}{\sqrt{Dt}}\right)}$$

Defining $s = x/\sqrt{Dt}$ so $u(x,t) = u_0 F(s)$ then compute u_t, u_{xx}:

$$(5) \qquad u_t = u_0 F'(s) \left(-\frac{x}{2\sqrt{D}} t^{-3/2}\right)$$

$$(6) \qquad u_{xx} = u_0 F''(s) \left(\frac{1}{Dt}\right)$$

Substitute (5)-(6) in $u_t = D u_{xx}$. $F(s)$ must satisfy

$$F''(s) = -\frac{1}{2} s F'(s)$$

whose general solution is $\sqrt{\pi}$

$$F(s) = c_1 \int_0^s e^{-\tau^2/4} d\tau + c_2$$

Pick c_k so (2)-(3) boundary conditions satisfied:

$$(7) \qquad u(x,t) = u_0 \left(1 - \frac{2}{\sqrt{\pi}} \int_0^{x/\sqrt{4Dt}} e^{-\tau^2} d\tau\right)$$