Singular Perturbation Theory - Algebraic

Some solutions of

$$f(x, \varepsilon) = 0 \tag{1}$$

may not have regular expansions

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots \tag{2}$$

but may be expressable using different guage functions. We illustrate this by way of example

EXAMPLE $\quad \varepsilon x^2 - 1 + \varepsilon = 0$

No solution of this equation has a regular expansion like (2). If we assumed (2) we would arrive at the contradictory statement $-1 = 0$, from which we would conclude our assumption was false.

Here we know the exact solution $\bar{x}(\varepsilon)$

$$\bar{x}(\varepsilon) = \pm\sqrt{\frac{1-\varepsilon}{\varepsilon}}$$

which we can expand

$$\bar{x}(\varepsilon) = \pm\, \varepsilon^{-1/2}(1-\varepsilon)^{1/2}$$

$$= \pm\, \varepsilon^{-1/2}\left(1 - \tfrac{1}{2}\varepsilon - \tfrac{1}{8}\varepsilon^2 + \cdots\right)$$

$$= \pm\left(\frac{1}{\sqrt{\varepsilon}} - \frac{1}{2}\sqrt{\varepsilon} - \frac{1}{8}\varepsilon^{3/2} + \cdots\right)$$

which is singular (grows) in ε and appropriate guage functions would be $\varepsilon^{n/2}$, $n = -1, 0, 1, 2 \ldots$

Thus, it might be reasonable to expect different kinds of asymptotic expansions for different roots.

For many problems involving powers of x, the solution is often expressable in powers of

$$\delta(\varepsilon) = \varepsilon^{\beta}$$

for some β, i.e. $\beta = \frac{1}{3}, \frac{1}{2}$ as in

$$\varepsilon x^3 - 1 = 0 \quad , \quad x = \varepsilon^{-1/3}$$

$$\varepsilon x^2 - 1 = 0 \quad , \quad x = \varepsilon^{-1/2}$$

Both these latter examples have <u>singular behavior</u> in that they grow as $\varepsilon \to 0$.

Some solutions may not be regular, involve different guage functions but are not singular. A trivial example:

$$x^2 - \varepsilon = 0 \quad , \quad x = \pm\sqrt{\varepsilon}$$

<u>Rescaling and Dominant Balance</u>

A common way to find singular solutions of

$$f(x, \varepsilon) = 0$$

is to scale x by $\mu(\varepsilon) = \varepsilon^{\alpha}$ as follows

$$F(X, \varepsilon) = f(\varepsilon^{\alpha} X, \varepsilon) \quad , \quad X = \frac{x}{\varepsilon^{\alpha}}$$

and the seek a series solution for X in an appropriate power of ε.

EXAMPLE $\boxed{f(x,\varepsilon) = \varepsilon x^3 - x + \varepsilon = 0}$

First note that there is a regular soln

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3)$$

From standard methods $x_0 = 0,\ x_1 = 1,\ x_2 = 0$
so that

$$x = \varepsilon + O(\varepsilon^3)$$

is an approximation of that regular soln.

Seek large singular root now.

$$(1) \qquad x = \frac{X}{\varepsilon^{\alpha}} \qquad \alpha > 0$$

then $f(x,\varepsilon)$ becomes

$$\varepsilon^{1-3\alpha} X^3 - \varepsilon^{-\alpha} X + \varepsilon = 0$$

$$(2) \qquad \underset{①}{X^3} - \underset{②}{\varepsilon^{2\alpha-1} X} + \underset{③}{\varepsilon^{3\alpha}} = 0 \qquad ② \gg ③$$

Choose α so that terms ① and ② are of the same order, i.e,

$$2\alpha - 1 = 0$$

$$\boxed{\alpha = \tfrac{1}{2}}$$

yields

$$(3) \qquad X^3 - X + \varepsilon^{3/2} = 0$$

Note then that $f(x,\varepsilon)=0 \iff F(\underline{X},\delta)$ where

$$(4) \qquad F(\underline{X},\delta) = \underline{X}^3 - \underline{X} + \delta = 0$$

where $\delta = \varepsilon^{3/2} \ll 1$.

The problem (4) has a regular expansion in δ, i.e.

$$\underline{X} = \underline{X}_0 + \delta \underline{X}_1 + O(\delta^2)$$

yields

$$O(1) \qquad \underline{X}_0^3 - \underline{X}_0 = 0$$

$$O(\delta) \qquad 3\underline{X}_0^2 \underline{X}_1 - \underline{X}_1 = -1$$

Consider only nonzero solns of $O(1)$ problem

$$\underline{X}_0 = \pm 1$$

regardless of sign$(\underline{X}_0)$ we have

$$\underline{X}_1 = -\tfrac{1}{2}$$

and

$$\underline{X} = \pm 1 - \tfrac{1}{2}\delta(\varepsilon) + O(\delta^2)$$

In terms of original scaling the two large roots are

$$x = \frac{1}{\sqrt{\varepsilon}}\left(\pm 1 - \tfrac{1}{2}\varepsilon^{3/2} + O(\varepsilon^3)\right)$$

EXAMPLE $\boxed{f(x,\varepsilon) = \varepsilon x^4 - x - 1 = 0}$

A regular solution is (by standard methods)

$$(1) \qquad x = 1 - \varepsilon - 4\varepsilon^2 + O(\varepsilon^3)$$

Seek singular solutions. Rescale x:

$$(2) \qquad x = \frac{X}{\varepsilon^\alpha} \qquad \alpha > 0$$

where α is to be chosen. Re-express equation in terms of X.

$$\varepsilon^{1-4\alpha} X^4 - \varepsilon^{-\alpha} X - 1 = 0$$

$$X^4 - \varepsilon^{3\alpha-1} X - \varepsilon^{4\alpha-1} = 0$$

$$1 \qquad 2 \gg 3$$

Choose α so that terms 1 and 2 have same order

$$(3) \qquad \alpha = 1/3$$

yields

$$(4) \qquad X^4 - X - \varepsilon^{1/3} = 0$$

Seek expansion of (4) of the form

$$X = X_0 + \delta(\varepsilon) X_1 + O(\delta^2)$$

where

$$\delta(\varepsilon) = \varepsilon^{1/3}$$

Obtain $O(1)$ and $O(\delta)$ equations

$$O(1) \qquad \bar{X}_0^4 - \bar{X}_0 = 0$$

$$O(\delta) \qquad 4\bar{X}_0^3 \bar{X}_1 - \bar{X}_1 - 1 = 0$$

whose soln (non zero, real) is

$$\bar{X}_0 = 1 \qquad \bar{X}_1 = \tfrac{1}{3}$$

Hence

$$\bar{X} = 1 + \tfrac{1}{3}\delta(\varepsilon) + O(\delta^2)$$

Given (2) we obtain the following expansion for the singular (large) root of $f(x,\varepsilon)=0$.

$$(5) \qquad x = \frac{1}{\varepsilon^{1/3}}\left(1 + \tfrac{1}{3}\varepsilon^{1/3} + O(\varepsilon^{2/3})\right)$$

Expanded out

$$x = \frac{1}{\varepsilon^{1/3}} + \frac{1}{3} + O(\varepsilon^{1/3})$$

<u>Remarks</u>: In summary we found two real roots for the quartic. The other two roots are complex.

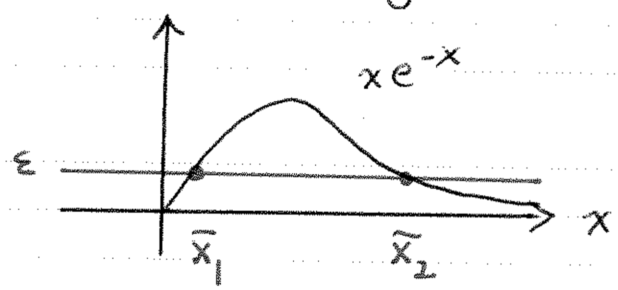
Of the two real roots the regular soln is $O(1)$ and the singular soln is $O(\varepsilon^{-1/3})$

Dominant Balance (Last Comments)

Asymptotic expansions of roots need not be in powers of ε. Consider for instance

$$(1) \qquad f(x,\varepsilon) = x e^{-x} - \varepsilon = 0$$

has two roots $\bar{x}_1(\varepsilon)$ and $\bar{x}_2(\varepsilon)$ as can be seen from the graph



A regular expansion yields

$$\bar{x}_1(\varepsilon) = \varepsilon + \varepsilon^2 + \tfrac{3}{2}\varepsilon^3 + O(\varepsilon^4)$$

For the large root one must assume

$$\bar{x}_2(\varepsilon) = x_0\,\mu(\varepsilon) + o(\mu)$$

Taking log of (1)

$$\log x - x - \log \varepsilon = 0$$

$$\underset{①}{\log(x_0\mu + o(\mu))} - \underset{②}{x_0\mu} - \underset{③}{\log\varepsilon} + o(\mu) = 0$$

If one chooses $\mu(\varepsilon) = \ln\varepsilon$ then one can show for $x_0 = -1$ the term ① is $o(\mu)$ as well.

One then finds

$$\bar{x}_2(\varepsilon) = -\ln\varepsilon + o(\ln\varepsilon)$$

Through a procedure called "bootstrapping" one can continue to expand $\bar{x}_2(\varepsilon)$. In this case (without details)

$$(2) \quad \bar{x}_2(\varepsilon) = -\ln\varepsilon + \ln|\ln\varepsilon| + o(\ln|\ln\varepsilon|)$$

The point:

Some algebraic equations can be represented as powers of ε, i.e.

$$\bar{x}(\varepsilon) = \varepsilon^{-\alpha}(x_0 + x_1\varepsilon^{\beta} + x_2\varepsilon^{2\beta} + \cdots)$$

Here $\alpha = 0$ gives the regular series and if $\alpha > 0$ the root is singular.

However, some roots have expansions involving unusual guage functions such as that (singular) root in eqn (2) above.