Singular Perturbation Theory - Algebraic

Some solutions of

\( f(x, \varepsilon) = 0 \)

may not have regular expansions

\( x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots \)

but may be expressable using different gauge functions. We illustrate this by way of example

**EXAMPLE** \( \varepsilon x^2 - 1 + \varepsilon = 0 \)

No solution of this equation has a regular expansion like (2). If we assumed (2) we would arrive at the contradictory statement \(-1 = 0\), from which we would conclude our assumption was false.

Here we know the exact solution \( \bar{x}(\varepsilon) \)

\[ \bar{x}(\varepsilon) = \pm \sqrt{\frac{1 - \varepsilon}{\varepsilon}} \]

which we can expand

\[ \bar{x}(\varepsilon) = \pm \varepsilon^{-\frac{1}{2}} (1 - \varepsilon)^{\frac{1}{2}} \]

\[ = \pm \varepsilon^{-\frac{1}{2}} (1 - \frac{1}{2} \varepsilon - \frac{1}{8} \varepsilon^2 + \cdots) \]

\[ = \pm \left( \frac{1}{\sqrt{\varepsilon}} - \frac{1}{2 \sqrt{\varepsilon}} - \frac{1}{8} \varepsilon^{\frac{3}{2}} + \cdots \right) \]

which is singular (grows) in \( \varepsilon \) and appropriate gauge functions would be \( \varepsilon^{\frac{n}{2}}, n = -1, 0, 1, 2, \ldots \)
Thus, it might be reasonable to expect different kinds of asymptotic expansions for different roots.

For many problems involving powers of $x$, the solution is often expressible in powers of

$$S(\varepsilon) = \varepsilon^{\beta}$$

for some $\beta$, i.e. $\beta = \frac{1}{3}, \frac{1}{2}$ as in

$$\varepsilon x^3 - 1 = 0 \quad \Rightarrow \quad x = \varepsilon^{-\frac{1}{3}}$$

$$\varepsilon x^2 - 1 = 0 \quad \Rightarrow \quad x = \varepsilon^{-\frac{1}{2}}$$

Both these latter examples have singular behavior in that they grow as $\varepsilon \to 0$.

Some solutions may not be regular, involve different gauge functions but are not singular. A trivial example:

$$x^2 - \varepsilon = 0 \quad \Rightarrow \quad x = \pm \sqrt{\varepsilon}$$

Re-scaling and Dominant Balance

A common way to find singular solutions of

$$f(x, \varepsilon) = 0$$

is to scale $x$ by $\mu(\varepsilon) = \varepsilon^\alpha$ as follows

$$F(\bar{x}, \varepsilon) = f(\varepsilon^\alpha \bar{x}, \varepsilon) \quad \Rightarrow \quad \bar{x} = \frac{x}{\varepsilon^\alpha}$$

and the seek a series solution for $\bar{x}$ in an appropriate power of $\varepsilon$. 
**Example**

\[ f(x, \varepsilon) = \varepsilon x^3 - x + \varepsilon = 0 \]

First note that there is a regular soln

\[ x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3) \]

From standard methods, \( x_0 = 0, x_1 = 1, x_2 = 0 \)

so that

\[ x = \varepsilon + O(\varepsilon^3) \]

is an approximation of that regular soln.

Seek large singular root now.

(1) \[ x = \frac{X}{\varepsilon^\alpha} \quad \alpha > 0 \]

then \( f(x, \varepsilon) \) becomes

\[ \varepsilon^{1-3\alpha} X^3 - \varepsilon^{-\alpha} X + \varepsilon = 0 \]

(2) \[ X^3 - \varepsilon^{2\alpha-1} X + \varepsilon^{3\alpha} = 0 \]

(1) \( \Rightarrow \) (2)

Choose \( \alpha \) so that terms (1) and (2) are of the same order, i.e.,

\[ 2\alpha - 1 = 0 \]

\[ \alpha = \frac{1}{2} \]

yields

(3) \[ X^3 - X + \varepsilon^{3/2} = 0 \]
Note then that \( f(x, \varepsilon) = 0 \iff F(x, \delta) \)
where
\[
(4) \quad F(x, \delta) = x^3 - x + \delta = 0
\]
where \( \delta = \varepsilon^{3/2} \ll 1 \).

The problem (4) has a regular expansion in \( \delta \), i.e.
\[
x = x_0 + \delta x_1 + O(\delta^2)
\]
yields

\[
O(1) \quad x_0^3 - x_0 = 0
\]
\[
O(\delta) \quad 3x_0^2 x_1 - x_1 = -1
\]

Consider only non-zero solutions of \( O(1) \) problem
\[
x_0 = \pm 1
\]
regardless of \( \text{sign}(x_0) \) we have
\[
x_1 = -\frac{1}{2}
\]
and
\[
x = \pm 1 - \frac{1}{2} \delta + O(\delta^2)
\]
In terms of original scaling the two large roots are
\[
x = \frac{1}{\sqrt{\varepsilon}} \left( \pm 1 - \frac{1}{2} \varepsilon^{3/2} + O(\varepsilon^3) \right)
\]
EXAMPLE

\[ f(x, \varepsilon) = \varepsilon x^4 - x - 1 = 0 \]

A regular solution is (by standard methods)

(1) \[ x = 1 - \varepsilon - 4\varepsilon^2 + O(\varepsilon^3) \]

Seek singular solutions. Rescale \( x \):

(2) \[ x = \frac{X}{\varepsilon^\alpha} \quad \alpha > 0 \]

where \( \alpha \) is to be chosen. Re-express equation in terms of \( X \):

\[ \varepsilon^{1-4\alpha} X^4 - \varepsilon^{-\alpha} X - 1 = 0 \]

\[ X^4 - \varepsilon^{3\alpha-1} X - \varepsilon^{4\alpha-1} = 0 \]

(1) \( \Rightarrow \) (2) \( \Rightarrow \) (3)

Choose \( \alpha \) so that terms (1) and (2) have same order.

(3) \[ \alpha = \frac{1}{3} \]

yields

(4) \[ X^4 - X - \varepsilon^{\frac{1}{3}} = 0 \]

Seek expansion of (4) of the form

\[ X = X_0 + \delta(\varepsilon) X_1 + O(\delta^2) \]

where

\[ \delta(\varepsilon) = \varepsilon^{\frac{1}{3}} \]
Obtain $O(1)$ and $O(\varepsilon)$ equations

$O(1)$ \quad \bar{X}_0^4 - \bar{X}_0 = 0$

$O(\varepsilon)$ \quad 4\bar{X}_0^3\bar{X}_1 - \bar{X}_1 - 1 = 0$

whose soln (non-zero, real) is

\[\bar{X}_0 = 1, \quad \bar{X}_1 = \frac{1}{3}\]

Hence

\[\bar{X} = 1 + \frac{1}{3}\varepsilon + O(\varepsilon^2)\]

Given (2) we obtain the following expansion for the singular (large) root of $f(x, \varepsilon) = 0$.

(5) \quad x = \frac{1}{\varepsilon^{\frac{1}{3}}} \left( 1 + \frac{1}{3} \varepsilon^{\frac{1}{3}} + O(\varepsilon^{\frac{2}{3}}) \right)

Expanded out

\[x = \frac{1}{\varepsilon^{\frac{1}{3}}} + \frac{1}{3} + O(\varepsilon^{\frac{1}{3}})\]

Remarks: In summary we found two real roots for the quartic. The other two roots are complex.

Of the two real roots the regular soln is $O(1)$ and the singular soln is $O(\varepsilon^{-\frac{1}{3}})$
Dominant Balance (Last Comments)

Asymptotic expansions of roots need not be in powers of $\varepsilon$. Consider for instance

$$f(x, \varepsilon) = x e^{-x} - \varepsilon = 0$$

has two roots $\overline{x}_1(\varepsilon)$ and $\overline{x}_2(\varepsilon)$ as can be seen from the graph.

A regular expansion yields

$$\overline{x}_1(\varepsilon) = \varepsilon + \varepsilon^2 + \frac{3}{2} \varepsilon^3 + O(\varepsilon^4)$$

For the large root one must assume

$$\overline{x}_2(\varepsilon) = x_0 \mu(\varepsilon) + o(\mu)$$

Taking log of (1)

$$\log x - x - \log \varepsilon = 0$$

$$\log (x_0 \mu + o(\mu)) - x_0 \mu - \log \varepsilon + o(\mu) = 0$$

(1) (2) (3)

If one chooses $\mu(\varepsilon) = \ln \varepsilon$ then one can show for $x_0 = -1$ the term (1) is $o(\mu)$ as well.
One then finds

\[ \overline{x}_2(\varepsilon) = -\ln \varepsilon + o(\ln \varepsilon) \]

Through a procedure called "bootstrapping" one can continue to expand \( \overline{x}_2(\varepsilon) \).
In this case (without details)

\[ (2) \quad \overline{x}_2(\varepsilon) = -\ln \varepsilon + \ln |\ln \varepsilon| + o(\ln |\ln \varepsilon|) \]

The point:

Some algebraic equations can be represented as powers of \( \varepsilon \), i.e.

\[ \overline{x}(\varepsilon) = \varepsilon^{-\delta} \left( x_0 + x_1 \varepsilon + x_2 \varepsilon^2 + \ldots \right) \]

Here \( \delta = 0 \) gives the regular series and if \( \delta > 0 \) the root is singular.

However, some roots have expansions involving unusual gauge functions such as that (singular) root in eqn (2) above.