Boundary Value Problems - Introductory example

A Boundary Value Problem (BVP) consists of a differential equation and a set of boundary conditions where the value of $y(x)$ and/or its derivatives are specified at endpoints:

EXAMPLE

(1) $$y'' - y = 0 \qquad x \in (0,1)$$

(2) $$y(0) = 2 \quad , \quad y(1) = 2e^{-1}$$

Solution: The general solution of (1) is

$$y(x) = c_1 e^{x} + c_2 e^{-x}$$

for some unknown constants c_1 and c_2. These are found using the boundary conditions

(3) $$y(0) = c_1 + c_2 = 2$$

(4) $$y(1) = e\,c_1 + e^{-1} c_2 = 2e^{-1}$$

Eqns (3)-(4) are simultaneous equations for unknowns c_1, c_2. Solving them one finds

$$c_1 = 0 \qquad c_2 = 2$$

so that the solution of the BVP (1)-(2) is

$$y(x) = 2e^{-x}$$

Boundary Layers (overview)

For small $\varepsilon > 0$ the following (linear) boundary value problem

$$(1) \qquad \varepsilon y'' + a(x,\varepsilon)\, y' + b(x,\varepsilon)\, y = f(x,\varepsilon)$$

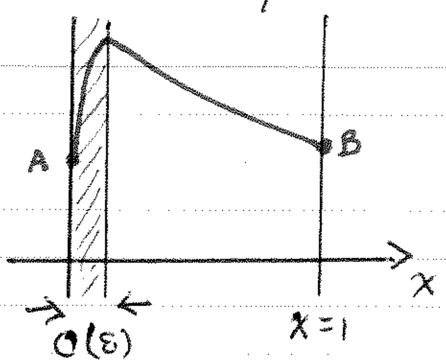
$$(2) \qquad y(0,\varepsilon) = A \qquad\qquad y(1,\varepsilon) = B$$

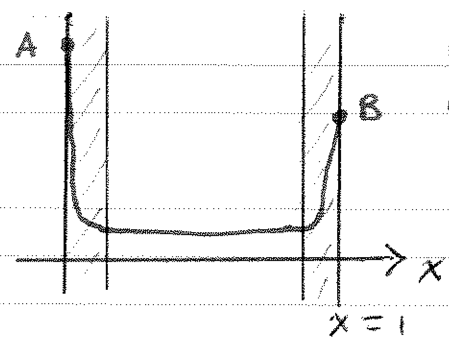
may have solutions exhibiting "layer" behavior.

This layer behavior occurs at x values where the $\varepsilon y''$ term is not small. Regular perturbation techniques don't normally yield good approximations.

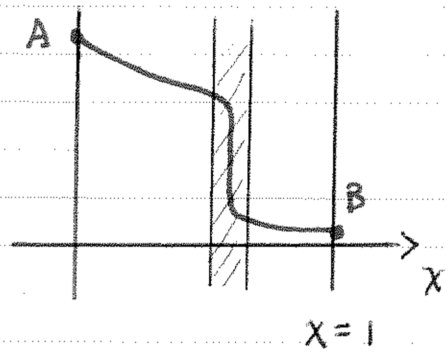
Types of Layers

Boundary layer

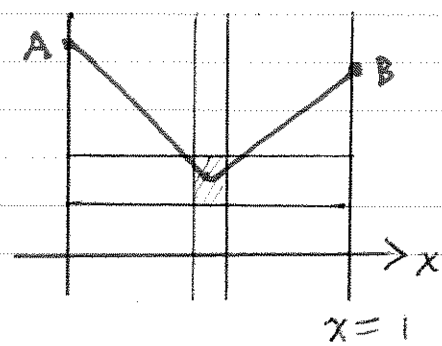




multiple boundary layer.



interior layer



corner layer

Model Problem

$$\boxed{\begin{array}{ll} (1) & \varepsilon y'' + y' + y = 0 \\ (2) & y(0,\varepsilon) = 4 \quad , \quad y(1,\varepsilon) = 5 \end{array}}$$

has a known exact solution which, despite the simplicty of the equation, is <u>very complicated</u>

$$y(x,\varepsilon) = c_+(\varepsilon)\, e^{\lambda_+(\varepsilon)x} + c_-(\varepsilon)\, e^{\lambda_-(\varepsilon)x}$$

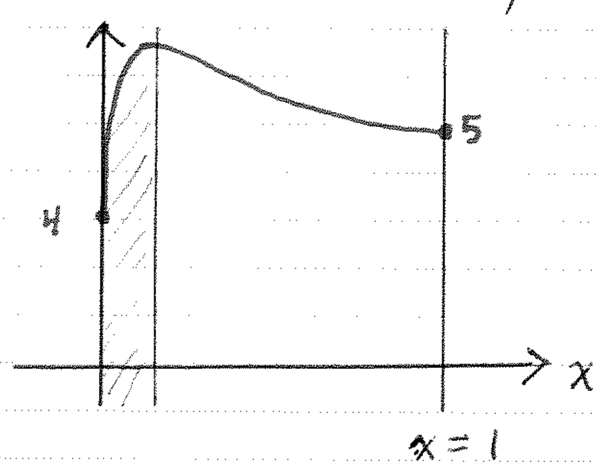
where

$$\lambda_\pm(\varepsilon) = \frac{-1 \pm \sqrt{1-4\varepsilon}}{2\varepsilon}$$

$$c_+(\varepsilon) = -\frac{(4e^{\lambda_-(\varepsilon)} - 5)}{e^{\lambda_+(\varepsilon)} - e^{\lambda_-(\varepsilon)}}$$

$$c_-(\varepsilon) = +\frac{(4e^{\lambda_+(\varepsilon)} - 5)}{(e^{\lambda_+(\varepsilon)} - e^{\lambda_-(\varepsilon)})}$$

For small $\varepsilon > 0$ this exact solution has a narrow boundary layer near $x=0$



In this layer y' is large and $\varepsilon y''$ is not small.

Derivation of the exact soln

Letting $y = e^{\lambda x}$ one obtains the characteristic equation

$$\varepsilon \lambda^2 + \lambda + 1 = 0$$

whose roots are $\lambda_\pm(\varepsilon)$. The general Solution is

$$y(x,\varepsilon) = c_+ e^{\lambda_+ x} + c_- e^{\lambda_- x}$$

for some (ε-dependent) constants $c_\pm$.

To satisfy the boundary conditions $c_\pm$ must satisfy

$$y(0,\varepsilon) = c_+ + c_- = 4$$
$$y(1,\varepsilon) = c_+ e^{\lambda_+} + c_- e^{\lambda_-} = 5$$

which are two eqns for two unknowns:

$$\begin{bmatrix} 1 & 1 \\ e^{\lambda_+} & e^{\lambda_-} \end{bmatrix} \begin{bmatrix} c_+ \\ c_- \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

Solving this system yields previously stated values $c_\pm(\varepsilon)$.

Outer Expansion of Exact Solution

Seek an asymptotic expansion of exact solution $y(x,\varepsilon)$ valid for fixed $x>0$.

Can show (after some work)

$$\lambda_+(\varepsilon) \sim -1 + O(\varepsilon)$$

$$\lambda_-(\varepsilon) \sim -\tfrac{1}{\varepsilon} + O(1)$$

$$c_+(\varepsilon) \sim 5e + o(1)$$

$$c_-(\varepsilon) \sim -5e+4 + o(1)$$

Recalling $y(x,\varepsilon) = c_+ e^{\lambda_+ x} + c_- e^{\lambda_- x}$, so long as $x \neq 0$

$$y(x,\varepsilon) = (5e + o(1))\, e^{(-1+O(\varepsilon))x} + (4-5e + o(1)) \underbrace{e^{(-\frac{1}{\varepsilon}+O(1))x}}_{\ll 1 \text{ for } x>0}$$

Hence

$$y(x,\varepsilon) = 5e \cdot e^{-x} + o(1)$$

$$y(x,\varepsilon) = 5e^{1-x} + o(1)$$

$$y(x,\varepsilon) = \underset{\Uparrow}{y_0(x)} + o(1)$$

leading order outer expansion

Note that while $y_0(x) = 5e^{1-x}$ satisfies one boundary condition $y_0(1)=5$ it is badly off at $x=0$

$$y_0(0) = 5e \neq 4 = y(0,\varepsilon) \quad !!$$

Outer Expansion not knowing exact solution

Assume

$$(1) \qquad y(x,\varepsilon) = y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2)$$

Using this in the differential equation

$$\varepsilon y'' + y' + y = 0$$

we conclude $y_0(x)$ is a solution of

$$O(1) \qquad y_0' + y_0 = 0$$

whose general solution is $y_0(x) = Ae^{-x}$.

QUESTION: which boundary condition does $y_0(x)$ satisfy? Suppose we assume correctly that

$$y_0(1) = 5$$

Then $y_0(1) = Ae^{-1} = 5 \Rightarrow A = 5e$ so that

$$y_0(x) = 5e^{1-x}$$

which is the same as that knowing the exact soln apriori.

Key Point The expansion

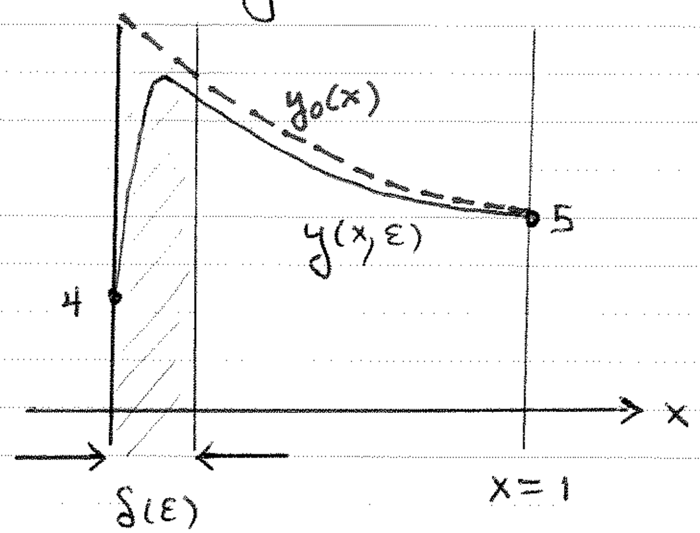
$$y(x,\varepsilon) = y_0(x) + O(\varepsilon)$$

yields a good approximation for $x > 0$ but fails badly near $x = 0$ since

$$y_0(0) = 5e \neq 4 = y(0,\varepsilon)$$

Inner Expansion (Boundary Layer)

The leading order outer approximation $y_0(x)$ well approximates the exact solution $y(x,\varepsilon)$ so long as $x = O(1)$



Seek a new "inner" approximation valid in the boundary layer of thickness $O(\delta)$

Toward this end we change variables

$$y(x,\varepsilon) = Y(X,\varepsilon) \qquad X = \frac{x}{\delta(\varepsilon)}$$

Notice that $X = O(1) \iff x = \delta(\varepsilon)$ is small.

The chain rule yields

$$\frac{dy}{dx} = \frac{dY}{dX}\frac{dX}{dx} = \frac{1}{\delta}\frac{dY}{dX}$$

$$\frac{d^2y}{dx^2} = \frac{1}{\delta^2}\frac{d^2Y}{dX^2}$$

Assuming

$$Y(X,\varepsilon) = Y_0(X) + \varepsilon Y_1(X) + O(\varepsilon^2)$$

we find the leading order inner problem

(5) $\quad Y_0'' + Y_0' = 0 \qquad Y_0(0) = 4$

whose solution is

(6) $\quad Y_0(X) = 4 + A(1 - e^{-X})$

for some (as yet) unknown constant A.

We determine "A" in the next section using a procedure called matching

Thus

$$(1) \qquad \varepsilon y'' + y' + y = 0$$

becomes

$$(2) \qquad \frac{\varepsilon}{\delta^2}\frac{d^2 Y}{d X^2} + \frac{1}{\delta}\frac{dY}{dX} + Y = 0$$

$$① \sim ② \gg ③$$

Here ② $\gg$ ③ for any $\delta \ll 1$. Terms ① and ② are of the same order only if

$$\frac{\varepsilon}{\delta^2} = \frac{1}{\delta}$$

yields the boundary layer thickness

$$\delta(\varepsilon) = \varepsilon$$

and equation (2) becomes

$$(3) \qquad Y'' + Y' + \varepsilon Y = 0$$

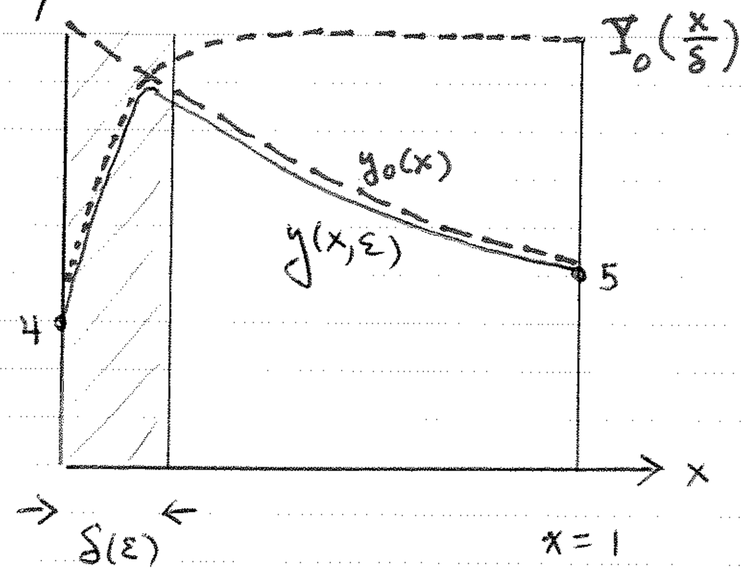
Want the solution to satisfy the left boundary condition since is s.

$$(4) \qquad y(0,\varepsilon) = Y(0,\varepsilon) = 4$$

Collectively (3)-(4) is the inner problem.

Matching

The nonrigorous idea of "matching" is that the outer solution going into the layer should "match" or equal the inner solution leaving the layer.



Since

$$\Upsilon = \frac{x}{\delta(\varepsilon)} \qquad \delta(\varepsilon) \ll 1$$

then $\Upsilon \to +\infty$ as $\varepsilon \to 0$ for $x > 0$.

Thus this casual idea of matching has a more rigorous form

$$M = \lim_{x \to 0^+} y_0(x) = \lim_{\Upsilon \to \infty} \Upsilon_0(\Upsilon)$$

This is also called Prandtl Matching and can be made more rigorous.

Recall for our problem

(1) $\varepsilon y'' + y' + y = 0$

(2) $y(0) = 4 \qquad y(1) = 5$

we found the following leading outer and inner approximations:

$$y_0(x) = 5e^{1-x}$$

$$Y_0(X) = 4 + A(1 - e^{-X})$$

Matching requires

$$M = \lim_{x \to 0^+} y_0(x) = \lim_{X \to \infty} Y_0(X)$$

$$M = 5e = 4 + A$$

Solving for $A = 5e - 4$ we can now complete the inner approximation

$$Y_0(X) = 4 + (5e-4)(1 - e^{-X})$$

Uniformly valid approximation

While $y_0(x)$ well approximates $y(x,\varepsilon)$ for $x > 0$ it fails miserably near $x=0$. Conversely, while $Y_0(X)$ well approximation of $y(x,\varepsilon)$ in the layer it is a poor approximation outside the layer.

Seek a single uniformly valid approximation on entire interval:

$$(1)\quad y_u(x,\varepsilon) \equiv y_0(x) + Y_0\left(\frac{x}{\delta}\right) - M$$

where M is the matching term

$$M = \lim_{x\to 0^+} y_0(x) = \lim_{X\to\infty} Y_0(X)$$

Why does this work?!

For fixed x, $\varepsilon \to 0^+$ we have $\frac{x}{\delta} \to \infty$ hence

$$y_u(x,\varepsilon) \sim y_0(x) + \cancel{M} - \cancel{M}$$

For fixed X, $\varepsilon \to 0^+$ we have

$$y_u(x,\varepsilon) = y_0(\delta X) + Y_0(X) - M$$

$$\sim \cancel{M} + Y_0(X) - \cancel{M}$$

For our model problem

$$y_0(x) = 5e^{1-x} \qquad Y_0(X) = 4 + (5e-4)(1-e^{-X})$$

we have $M = y_0(0^+) = 5e$ so the uniformly valid approximation is

$$y_u(x,\varepsilon) = \underbrace{5e^{1-x}}_{y_0(x)} + \underbrace{4 + (5e-4)(1-e^{-x/\varepsilon})}_{Y_0(\frac{x}{\varepsilon})} - \underbrace{5e}_{M}$$

This simplifies to

$$\boxed{y_u(x,\varepsilon) = 5e^{(1-x)} - 5e^{(1-x/\varepsilon)} + 4e^{-x/\varepsilon}}$$

How good is the approximation? Boundary Cond:

$$y_u(0,\varepsilon) = 4$$

$$y_u(1,\varepsilon) = 5 - \underbrace{5e^{(1-1/\varepsilon)}}_{\ll 1} + \underbrace{4^{-1/\varepsilon}}_{\ll 1} \sim 5 + o(1)$$

Satisfies boundary conditions asymptotically. And if

$$L(y) \equiv \varepsilon y'' + y' + y$$

we get

$$L(y_u) = 5e^{(1-x)}\varepsilon - 5e^{(1-x/\varepsilon)} + 4e^{-x/\varepsilon}$$

$$L(y_u) = O(\varepsilon) \qquad x \in (0,1)$$

i.e. satisfies diff eq to $O(\varepsilon)$.