Boundary Value Problems - Introductory example

A Boundary Value Problem (BVP) consists of a differential equation and a set of boundary conditions where the value of \( y(x) \) and/or its derivatives are specified at endpoints.

**EXAMPLE**

\begin{align*}
(1) \quad & y'' - y = 0 \quad x \in (0, 1) \\
(2) \quad & y(0) = 2, \ y(1) = 2e^{-1}
\end{align*}

Solution: The general solution of (1) is

\[ y(x) = c_1 e^x + c_2 e^{-x} \]

for some unknown constants \( c_1 \) and \( c_2 \). These are found using the boundary conditions

\begin{align*}
(3) \quad & y(0) = c_1 + c_2 = 2 \\
(4) \quad & y(1) = e c_1 + e^{-1} c_2 = 2e^{-1}
\end{align*}

Eqns. (3)-(4) are simultaneous equations for unknowns \( c_1, c_2 \). Solving them one finds

\[ c_1 = 0, \quad c_2 = 2 \]

so that the solution of the BVP (1)-(2) is

\[ y(x) = 2e^{-x} \]
Boundary Layers (overview)

For small \( \varepsilon > 0 \) the following (linear) boundary value problem

\[
\begin{align*}
(1) & \quad \varepsilon y'' + a(x, \varepsilon) y' + b(x, \varepsilon) y = f(x, \varepsilon) \\
(2) & \quad y(0, \varepsilon) = A \quad y(1, \varepsilon) = B
\end{align*}
\]

may have solutions exhibiting "layer" behavior.

This layer behavior occurs at \( x \) values where the \( \varepsilon y'' \) term is not small. Regular perturbation techniques don't normally yield good approximations.

Types of Layers.

- Boundary layer
- Multiple boundary layer
- Interior layer
- Corner layer
Model Problem

(1) \[ \varepsilon y'' + y' + y = 0 \]

(2) \[ y(0, \varepsilon) = 4, \quad y(1, \varepsilon) = 5 \]

has a known exact solution which, despite the simplicity of the equation, is very complicated

\[ y(x, \varepsilon) = c_+ \varepsilon \lambda_+ x + c_- \varepsilon \lambda_- x \]

where

\[ \lambda_\pm(\varepsilon) = -1 \pm \frac{1 - 4\varepsilon}{2\varepsilon} \]

\[ c_+(\varepsilon) = -\frac{4e^{\lambda_+(\varepsilon)} - 5}{e^{\lambda_+(\varepsilon)} - e^{\lambda_-(\varepsilon)}} \]

\[ c_-(\varepsilon) = +\frac{4e^{\lambda_-(\varepsilon)} - 5}{e^{\lambda_+(\varepsilon)} - e^{\lambda_-(\varepsilon)}} \]

For small \( \varepsilon > 0 \) this exact solution has a narrow boundary layer near \( x = 0 \)

In this layer \( y' \) is large and \( \varepsilon y'' \) is not small.
Derivation of the exact soln

Letting $y = e^{\lambda x}$ one obtains the characteristic equation

$$\epsilon \lambda^2 + \lambda + 1 = 0$$

whose roots are $\lambda_{\pm}(\epsilon)$. The general solution is

$$y(x, \epsilon) = c_+ e^{\lambda_+ x} + c_- e^{\lambda_- x}$$

for some ($\epsilon$-dependent) constants $c_\pm$.

To satisfy the boundary conditions $c_\pm$ must satisfy

$$y(0, \epsilon) = c_+ + c_- = 4$$
$$y(1, \epsilon) = c_+ e^{\lambda_+} + c_- e^{\lambda_-} = 5$$

which are two eqns for two unknowns:

$$
\begin{bmatrix}
1 & 1 \\
\epsilon \lambda_+ & \epsilon \lambda_-
\end{bmatrix}
\begin{bmatrix}
c_+ \\
c_-
\end{bmatrix} = 
\begin{bmatrix}
4 \\
5
\end{bmatrix}
$$

Solving this system yields previously stated values $c_\pm(\epsilon)$. 

Outer Expansion of Exact Solution

Seek an asymptotic expansion of exact solution \( y(x, \varepsilon) \) valid for fixed \( x > 0 \).

Can show (after some work)

\[
\lambda_+ (\varepsilon) \sim -1 + O(\varepsilon)
\]
\[
\lambda_- (\varepsilon) \sim -\frac{1}{\varepsilon} + O(1)
\]
\[
c_+ (\varepsilon) \sim 5\varepsilon + o(1)
\]
\[
c_- (\varepsilon) \sim -5\varepsilon + 4 + o(1)
\]

Recalling \( y(x, \varepsilon) = c_+ e^{\lambda_+ x} + c_- e^{\lambda_- x} \), so long as \( x \neq 0 \)

\[
y(x, \varepsilon) = (5\varepsilon + o(1)) e^{(-1 + O(\varepsilon)) x}
\]
\[
+ (4 - 5\varepsilon + o(1)) e^{(-\frac{1}{\varepsilon} + O(1)) x}
\]

Hence \( \ll 1 \) for \( x > 0 \)

\[
y(x, \varepsilon) = 5\varepsilon e^{-x} + o(1)
\]
\[
y(x, \varepsilon) = 5e^{1-x} + o(1)
\]
\[
y(x, \varepsilon) = y_0 (x) + o(1)
\]

leading order outer expansion

Note that while \( y_0 (x) = 5e^{1-x} \) satisfies one boundary condition \( y_0 (1) = 5 \) it is badly off at \( x = 0 \)

\[
y_0 (0) = 5e \neq 4 = y(0, \varepsilon)
\]
Outer Expansion not knowing exact solution

Assume

\begin{equation}
(1) \quad y(x, \varepsilon) = y_0(x) + \varepsilon y_1(x) + O(\varepsilon^2)
\end{equation}

Using this in the differential equation

\[ \varepsilon y'' + y' + y = 0 \]

we conclude \( y_0(x) \) is a solution of

\begin{equation}
(0) \quad y_0' + y_0 = 0
\end{equation}

whose general solution is \( y_0(x) = A e^{-x} \).

**QUESTION:** Which boundary condition does \( y_0(x) \) satisfy? Suppose we assume correctly that

\[ y_0(1) = 5 \]

Then \( y_0(1) = Ae^{-1} = 5 \implies A = 5e \) so that

\[ y_0(x) = 5e^{1-x} \]

which is the same as that knowing the exact soln apriori.

**Key Point:** The expansion

\[ y(x, \varepsilon) = y_0(x) + O(\varepsilon) \]

yields a good approximation for \( x > 0 \) but fails badly near \( x = 0 \) since

\[ y_0(0) = 5e \neq 4 = y_{10, \varepsilon} \]
Inner Expansion (Boundary Layer)

The leading order outer approximation \( y_0(x) \) well approximates the exact solution \( y(x, \varepsilon) \) so long as \( x = O(1) \).

Seek a new "inner" approximation valid in the boundary layer of thickness \( O(\delta) \).

Toward this end we change variables

\[
y(x, \varepsilon) = \varphi(\overline{x}, \varepsilon) \quad \overline{x} = \frac{x}{\delta(\varepsilon)}
\]

Notice that \( \overline{x} = O(1) \iff x = O(\delta) \) is small.

The chain rule yields

\[
\frac{dy}{dx} = \frac{d\varphi}{d\overline{x}} \frac{d\overline{x}}{dx} = \frac{1}{\delta} \frac{d\varphi}{d\overline{x}}
\]

\[
\frac{d^2y}{dx^2} = \frac{1}{\delta^2} \frac{d^2\varphi}{d\overline{x}^2}
\]
Assuming

\[ \Xi(\bar{x}, \varepsilon) = \Xi_0(\bar{x}) + \varepsilon \Xi_1(\bar{x}) + O(\varepsilon^2) \]

we find the leading order inner problem

(5) \[ \Xi_0'' + \Xi_0' = 0 \quad \Xi_0(0) = 4 \]

whose solution is

(6) \[ \Xi_0(\bar{x}) = 4 + A(1 - e^{-\bar{x}}) \]

for some (as yet) unknown constant \( A \).

We determine "A" in the next section using a procedure called matching.
Thus

(1) $\varepsilon y'' + y' + y = 0$

becomes

(2) $\frac{\varepsilon}{\delta^2} \frac{d^2\overline{y}}{dx^2} + \frac{1}{\delta} \frac{d\overline{y}}{dx} + \overline{y} = 0$

\[ \text{(1)} \sim \text{(2)} \gg \text{(3)} \]

Here \( \text{(2)} \gg \text{(3)} \) for any \( \delta \ll 1 \). Terms \( \text{\textcircled{1}} \) and \( \text{\textcircled{2}} \) are of the same order only if

$$\frac{\varepsilon}{\delta^2} = \frac{1}{\delta}$$

yields the boundary layer thickness

$$\delta(\varepsilon) = \varepsilon$$

and equation (2) becomes

(3) $\overline{Y}'' + \overline{Y}' + \varepsilon \overline{Y} = 0$

Want the solution to satisfy the left boundary condition since is s:

(4) \( y(0, \varepsilon) = \overline{Y}(0, \varepsilon) = 4 \)

Collectively (3)-(4) is the inner problem.
Matching

The nonrigorous idea of "matching" is that the outer solution going into the layer should "match" or equal the inner solution leaving the layer.

Since \( \overline{x} = \frac{x}{S(\varepsilon)} \), \( S(\varepsilon) \ll 1 \)
then \( \overline{x} \to +\infty \) as \( \varepsilon \to 0 \) for \( x > 0 \).

Thus this casual idea of matching has a more rigorous form

\[
M = \lim_{x \to 0^+} y_0(x) = \lim_{\overline{x} \to \infty} \frac{y_0(\overline{x})}{\overline{x}}
\]

This is also called Prandtl Matching and can be made more rigorous.
Recall for our problem

\[ (1) \quad \varepsilon y'' + y' + y = 0 \]
\[ (2) \quad y(0) = 4 \quad y(1) = 5 \]

we found the following leading outer and inner approximations:

\[ y_0(x) = 5e^{1-x} \]
\[ \overline{y}_0(x) = 4 + A(1-e^{-x}) \]

Matching requires

\[ M = \lim_{x \to 0^+} y_0(x) = \lim_{x \to \infty} \overline{y}_0(x) \]
\[ M = 5e = 4 + A \]

Solving for \( A = 5e - 4 \) we can now complete the inner approximation

\[ \overline{y}_0(x) = 4 + (5e-4)(1-e^{-x}) \]
Uniformly valid approximation

while \( y_0(x) \) well approximates \( y(x, \varepsilon) \) for \( x > 0 \) it fails miserably near \( x = 0 \). Conversely, while \( \Gamma_0(x) \) well approximation of \( y(x, \varepsilon) \) in the layer it is a poor approximation outside the layer.

Seek a single uniformly valid approximation on entire interval:

\[
(1) \quad y_u(x, \varepsilon) \equiv y_0(x) + \Gamma_0 \left( \frac{x}{\varepsilon} \right) - M
\]

where \( M \) is the matching term

\[
M = \lim_{x \to 0^+} y_0(x) = \lim_{x \to \infty} \Gamma_0(x)
\]

Why does this work?!

For fixed \( x \), \( \varepsilon \to 0^+ \) we have \( \frac{x}{\varepsilon} \to \infty \) hence

\[
y_u(x, \varepsilon) \sim y_0(x) + M - M
\]

For fixed \( x \), \( \varepsilon \to 0^+ \) we have

\[
y_u(x, \varepsilon) = y_0(\delta x) + \Gamma_0(x) - M
\sim M + \Gamma_0(x) - M
\]
For our model problem

\[ y'_0(x) = 5e^{-x} \quad \overline{y}_0(x) = 4 + (5e-4)(1-e^{-x}) \]

we have \( M = y'_0(0^+) = 5e \) so the uniformly valid approximation is

\[ y'_u(x, \varepsilon) = \frac{5e^{-x} + 4 + (5e-4)(1-e^{-\frac{x}{\varepsilon}})}{y'_0(x)} \overline{y}_0 \frac{x}{\varepsilon} \frac{M}{5e} = \frac{x}{\varepsilon} \]

This simplifies to

\[ y_u(x, \varepsilon) = 5e^{-x} - 5e^{-\frac{x}{\varepsilon}} + 4e^{-\frac{x}{\varepsilon}} \]

How good is the approximation? BoundaryCond:

\[ y'_u(0, \varepsilon) = 4 \]

\[ y'_u(1, \varepsilon) = 5 - 5e^{-\frac{1}{\varepsilon}} + \frac{4}{\varepsilon} \ll 1 \]

Satisfies boundary conditions asymptotically.

And if

\[ L(y) = \varepsilon y'' + y' + y \]

we get

\[ L(y'_u) = 5e^{-x} - 5e^{-\frac{x}{\varepsilon}} + 4e^{-\frac{x}{\varepsilon}} \]

\[ L(y_u) = O(\varepsilon) \quad x \in (0, 1) \]

i.e., satisfies diff eq to \( O(\varepsilon) \).