EXAMPLE

$$\boxed{\begin{array}{ll} \varepsilon y'' + (x+1)y' + y = 0 & x \in (0,1) \\ y(0) = 0 \qquad y(1) = 1 & \end{array}}$$

Outer expansion $y(x,\varepsilon) = y_0(x) + o(1)$ yields

(1) $(x+1)y_0' + y_0 = 0$

(2) $y_0(1) = 1$

where we have presumed a layer at $x=0$.
The solution of (1)-(2) is

$$y_0(x) = \frac{2}{1+x}$$

Inner expansion $y(x,\varepsilon) = Y_0(X) + o(1)$

$$y(x,\varepsilon) = Y(X,\varepsilon) \qquad X = \frac{x}{\delta} \qquad \delta \ll 1$$

differential equation for Y is

(3) $\frac{\varepsilon}{\delta^2} Y'' + \frac{1}{\delta}(1+\delta X)Y' + Y = 0$

highest order (largest) terms must balance

Conclude boundary layer thickness

$$\delta(\varepsilon) = \varepsilon$$

and (3) becomes

(4) $Y'' + (1+\varepsilon X)Y' + \varepsilon Y = 0$

Using $Y(X, \varepsilon) = Y_0(X) + o(1)$ in eqn (4)

(5) $\quad Y_0'' + Y_0' = 0 \qquad\qquad Y_0(0) = 0$

where we have used the left B.C. $y(0) = 0$.

The solution of (5) is

$$Y_0(X) = C\,(e^{-X} - 1)$$

<u>Matching</u>

$$M = \lim_{x \to 0^+} y_0(x) = \lim_{X \to +\infty} Y_0(X)$$

$$M = 2 = -C$$

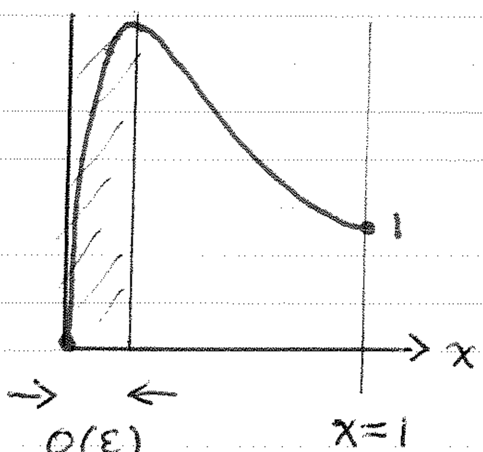
Hence $C = -2$ and the completed inner soln is:

$$Y_0(X) = 2(1 - e^{-X})$$

<u>Uniformly valid approximation</u>

$$y_u(x, \varepsilon) = y_0(x) + Y_0\left(\frac{x}{\varepsilon}\right) - M$$

$$y_u(x, \varepsilon) = \frac{2}{1 + x} - 2e^{-x/\varepsilon}$$



Exact soln is known but very messy.

EXAMPLE (Different B-Layer Thickness)

$$\boxed{\begin{array}{lll} (1) & \varepsilon y'' + x^{1/3} y' + y = 0 & \\ (2) & y(0) = 0 & y(1) = e^{-3/2} \end{array}}$$

Outer expansion $y(x,\varepsilon) = y_0(x) + o(1) \Rightarrow$

(3) $x^{1/3} y_0' + y_0 = 0 \quad , \quad y_0(1) = e^{-3/2}$

where a B-Layer at $x=0$ is presumed (explaining why outer y_0 satisfies B.C. at $x=1$). Soln of (3)

(4) $y_0(x) = e^{-\frac{3}{2}x^{2/3}}$

Inner soln

Let

$$y(x,\varepsilon) = Y(X,\varepsilon) \qquad X = \frac{x}{\delta} \quad , \quad \delta \ll 1$$

Substitute into diff eqn (1)

$$\frac{\varepsilon}{\delta^2} Y'' + \frac{\delta^{1/3}}{\delta} X^{1/3} Y' + Y = 0$$

multiply through by $\frac{\delta^2}{\varepsilon}$

(5) $Y'' + \frac{\delta^{4/3}}{\varepsilon} X^{1/3} Y' + \frac{\delta^2}{\varepsilon} Y = 0$

$$1 \sim 2 \gg 3$$

Must choose $\delta(\varepsilon)$ so $1 \sim 2$.

$$\delta(\varepsilon) = \varepsilon^{3/4} \qquad \text{B-Layer Thickness}$$

For this choice of δ, eqn (5) becomes

$$(6) \quad Y'' + X^{1/3} Y' + \varepsilon^{1/2} Y = 0$$

Note that although we only seek the leading order soln of (6), an appropriate higher order expansion would be

$$Y(X, \varepsilon) = Y_0(X) + \mu Y_1(X) + O(\mu^2)$$

where $\mu(\varepsilon) = \varepsilon^{1/2}$.

Leading inner expansion $Y = Y_0 + o(1) \Rightarrow$

$$(7) \quad Y_0'' + X^{1/3} Y_0' = 0, \quad Y_0(0) = 0$$

Integrating (7) once in X

$$Y_0'(X) = C_1 e^{-\frac{3}{4} X^{4/3}}$$

From which

$$(8) \quad Y_0(X) = C_1 \int_0^X e^{-\frac{3}{4} t^{4/3}} \, dt$$

defines Y_0 for some unknown C_1.

Now we match y_0 to Y_0.

Matching

$$M = \lim_{x \to 0} y_0(x) = \lim_{X \to \infty} Y_0(X)$$

$$(9) \quad M = 1 = C_1 \int_0^\infty e^{-\frac{3}{4}t^{4/3}} dt$$

Thus

$$(10) \qquad C_1 = \left(\int_0^\infty e^{-\frac{3}{4}t^{4/3}} dt \right)^{-1} \qquad \text{some constant.}$$

Knowing C_1 completes the inner approximation.

Uniformly valid solution

$$y_u(x, \varepsilon) = y_0(x) + Y_0\left(\frac{x}{\varepsilon^{3/4}}\right) - M$$

or

$$y_u(x, \varepsilon) = \left(e^{-\frac{3}{2}x^{2/3}} - 1 \right) + C_1 \int_0^{x/\varepsilon^{3/4}} e^{-\frac{3}{4}t^{4/3}} dt$$

where C_1 is defined by (10). The value of C_1 and the integral defining $y_u(x, \varepsilon)$ can be evaluated numerically.

EXAMPLE

$$\boxed{\varepsilon y'' + \frac{1}{1+x} y' + \varepsilon y = 0 \quad , \; y(0)=0, \; y(1) = 1}$$

Leading order outer problem (B-Layer at $x=0$)

$$\frac{1}{1+x} y_0' = 0 \qquad y_0(1) = 1$$

yields

$$y_0(x) = 1$$

Inner problem $\quad y = Y(X, \varepsilon), \quad X = \frac{x}{\delta}, \quad 0 < \delta \ll 1$

$$(1) \qquad \frac{\varepsilon}{\delta^2} Y'' + \frac{1}{(1+\delta X)} \frac{1}{\delta} Y' + \varepsilon Y = 0$$

$$① \sim ② \gg ③$$

Choice $\delta(\varepsilon) = \varepsilon$ makes ① $\sim$ ② and (1) becomes

$$Y'' + \frac{1}{1+\varepsilon X} Y' + \varepsilon^2 Y = 0$$

Leading problem / soln

$$Y_0'' + Y_0' = 0 \qquad Y_0(0) = 0$$

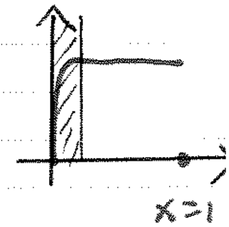
$$Y_0(X) = A(1 - e^{-X})$$

Matching

$$M = \lim_{x \to 0^+} y_0(x) = \lim_{X \to \infty} Y_0(X) = A = 1$$

Uniform soln

$$y_u(x, \varepsilon) = y_0(x) + Y_0\left(\frac{x}{\varepsilon}\right) - 1 = 1 - e^{-x/\varepsilon}$$



Known exact soln involves Bessel fns and is extremely complicated.

EXAMPLE

$$\boxed{\begin{array}{ll} \varepsilon y'' + y y' + y^3 = 0 \\ y(0) = 0 \qquad y(1) = 1 \end{array}}$$

nonlinear BVP

Outer expansion $y(x, \varepsilon) = y_0(x) + o(1)$

(1) $\quad y_0 y_0' + y_0^3 = 0 \qquad y_0(1) = 1$

where we have assumed a layer exists at $x = 0$. The solution of the IVP (1) is

$$y_0(x) = \frac{1}{x}$$

Inner expansion

$$y(x, \varepsilon) = Y(X, \varepsilon) \qquad X = \frac{x}{\delta(\varepsilon)} \qquad \delta \ll 1$$

then

$$\underset{①}{\frac{\varepsilon}{\delta^2} Y''} + \underset{②}{\frac{1}{\delta} Y Y'} + \underset{③}{Y^3} = 0$$

$$② \gg ③$$

Choose $\delta(\varepsilon) = \varepsilon$ so that ① ~ ② and then

$$Y'' + Y Y' + \varepsilon Y^3 = 0$$

For $Y(X, \varepsilon) = Y_0(X) + o(1)$ we find the leading inner problem

(2) $\quad Y_0'' + Y_0 Y_0' = 0 \qquad Y_0(0) = 0$

Solving this takes some work.

The soln (described in more detail at end)

$$\mathrm{Y}_0(\mathrm{X}) = \frac{1}{\alpha}\tanh\left(\frac{\mathrm{X}}{2\alpha}\right) \qquad \alpha > 0$$

where α is unknown. In particular $\mathrm{Y}_0(0) = 0$.

<u>Matching</u>

$$M = \lim_{x \to 0^+} y_0(x) = \lim_{\mathrm{X} \to \infty} \mathrm{Y}_0(\mathrm{X})$$

$$(3) \quad M = 1 = \frac{1}{\alpha}$$

where the latter is true since $\tanh z \to 1$ as $z \to +\infty$. Conclude $\alpha = 1$ and

$$\mathrm{Y}_0(\mathrm{X}) = \tanh\left(\frac{\mathrm{X}}{2}\right)$$

<u>Uniform Soln</u>

$$y_u(x, \varepsilon) = y_0(x) + \mathrm{Y}_0(\mathrm{X}) - M$$

$$(4) \quad y_u(x, \varepsilon) = \frac{1}{x} + \tanh\left(\frac{x}{2\varepsilon}\right) - 1$$

<u>Details on solution of inner problem</u>

Let $\mathrm{X} \to x$ and $\mathrm{Y}_0 \to$ so we seek soln of

$$(5) \quad \frac{d^2y}{dx^2} + y\frac{dy}{dx} = 0$$

Interchange dependent/independent variables

$$u(y) = \frac{dy}{dx}$$

From this

$$\frac{d^2y}{dx^2} = u'(y)\frac{dy}{dx} = u(y)\,u'(y)$$

Thus eqn (5) becomes

$$u\,u'(y) + y\,u = 0$$

$$u'(y) = -y$$

(6) $\quad \frac{dy}{dx} = u(y) = -\frac{1}{2}y^2 + c_1 \quad$ for some $c \in \mathbb{R}$

Eqn (6) is separable

(7) $\quad \frac{dy}{\frac{1}{2}y^2 - c_1} = -dx$

Using tables to integrate (7) we get

(8) $\quad \frac{\sqrt{2}}{\sqrt{c_1}}\,\mathrm{arctanh}\left(\frac{y\sqrt{2}}{2\sqrt{c_1}}\right) = x + c_2$

Invert

$$y(x) = \beta\tanh\left(\tfrac{1}{2}\beta(x+c_2)\right) \qquad \beta = -\sqrt{2c_1}$$

Here c_2, β are unknown (arbitrary) $\Rightarrow$ for $\Psi_0(0) = 0$

$$\Psi_0(X) = \frac{1}{\alpha}\tanh\left(\frac{X}{2\alpha}\right)$$