Failure in Matched asymptotics

When using the method of matched asymptotics to find approximate solutions to singular boundary value problems several things can go wrong.

Two common problems:

(a) Assuming a layer exists on the wrong side of the interval

(b) Picking an incorrect boundary layer thickness.

Ultimately in both these cases the method fails because the inner soln can't be matched to the outer soln.

Failure implies only the assumption about the layer location and/or thickness are wrong.

We illustrate these issues by way of example.

EXAMPLE (Incorrect layer location assumption)

$$(1) \qquad \varepsilon y'' + (x+1)y' + y = 0$$

$$(2) \qquad y(0) = 0 \qquad y(1) = 1$$

This problem was previously solved under the correct assumption there was a layer near $x=0$.

WHAT IF WE INCORRECTLY ASSUMED A LAYER EXISTED AT $x=1$?

Outer soln $\qquad y(x,\varepsilon) = y_0(x) + o(1)$

$$(3) \qquad (x+1)y_0' + y_0 = 0$$

$$(4) \qquad y_0(0) = 0$$

$\Leftarrow$ here since B-Lay is assumed to be at $x=1$, y_0 satisfies BC at $x=0$

The unique solution of (3)-(4)

$$y_0(x) = 0$$

i.e zero for all $x \in (0,1)$.

Inner soln

$$y(x,\varepsilon) = Y(X,\varepsilon) \qquad X = \frac{x-1}{\varepsilon}$$

where $\delta(\varepsilon) = \varepsilon$ is B-Lay thickness.
Note how X redefined for layer at $x=1$.

Diff Eqn for $\mathbb{Y}$ becomes

$$\mathbb{Y}'' + (2 + \varepsilon \mathbb{X}) \mathbb{Y}' + \varepsilon \mathbb{Y} = 0$$

For $\mathbb{Y}(\mathbb{X}, \varepsilon) = \mathbb{Y}_0(\mathbb{X}) + o(1)$ we find

$$\mathbb{Y}_0'' + 2\mathbb{Y}_0' = 0 \qquad \mathbb{Y}_0(0) = 1$$

where $\mathbb{Y}_0$ satisfies B.C. at $x=1$. Soln is

$$\mathbb{Y}_0(\mathbb{X}) = 1 + B(1 - e^{-2\mathbb{X}})$$

<u>Matching</u>

Since $\mathbb{X} = \frac{x-1}{\varepsilon}$ and $\varepsilon \to 0^+$, for fixed $x \in (0,1)$ $\mathbb{X} < 0$ and in fact $\mathbb{X} \to -\infty$ away from the boundary. Matching outer going into inner yields the matching condition

$$\lim_{x \to 1^-} y_0(x) = \lim_{\mathbb{X} \to -\infty} \mathbb{Y}_0(\mathbb{X})$$

$$(5) \qquad 0 = \lim_{\mathbb{X} \to -\infty} \left(1 + B(1 - e^{-2\mathbb{X}})\right)$$

There is no value of B to make the matching condition (5) true - even $B=0$ doesn't work!

<u>Conclude</u>: since the outer/inner expansions are unmatchable the assumption of the layer location at $x=1$ is false!

EXAMPLE (Incorrect layer thickness $\delta(\varepsilon)$)

(1) $$\varepsilon y'' + y' + y = 0$$

(2) $$y(0) = 0 \qquad y(1) = e^{-1}$$

Outer soln $y(x,\varepsilon) = y_0(x) + o(1)$

$$y_0' + y_0 = 0 \qquad y_0(1) = e^{-1}$$

yields $y_0(x) = e^{-x}$. Knowledge of outer soln doesn't impact choice of layer thickness.

Inner soln

$$y(x,\varepsilon) = Y(X,\varepsilon) \qquad X = \frac{x}{\delta(\varepsilon)}$$

where $0 < \delta(\varepsilon) \ll 1$. Sub into (1) yields

$$\frac{\varepsilon}{\delta^2} Y'' + \frac{1}{\delta} Y' + Y = 0$$

$$① \qquad ② \gg ③$$

Normally we would choose δ s.t. ① ~ ② which in this case would yield $\delta(\varepsilon) = \varepsilon$.

Suppose we chose $\delta(\varepsilon)$ differently.

Choice δ s.t. ① >> ② (such as $\delta(\varepsilon) = \varepsilon^2$)

$$\underline{Y}'' + \frac{\delta}{\varepsilon}\underline{Y}' + \frac{\delta^2}{\varepsilon}\underline{Y} = 0$$

① >> ② >> ③

then $\underline{Y} = \underline{Y}_0(\underline{X}) + o(1)$ yields

$$\underline{Y}_0'' = 0 \qquad\qquad \underline{Y}_0(0) = 0$$

or that $\underline{Y}_0(\underline{X}) = A\underline{X}$. But this can't be matched to the outer soln since there does not exist a constant A such that

$$\lim_{x \to 0^+} y_0(x) = 1 = \lim_{\underline{X} \to \infty} A\underline{X}$$

Choice δ s.t. ② >> ① (such as $\delta(\varepsilon) = \sqrt{\varepsilon}$)

$$\underline{Y}' + \frac{\varepsilon}{\delta}\underline{Y}'' + \delta\underline{Y} = 0$$

$\Uparrow$ Biggest of the three terms (dominant)

hence $\underline{Y} = \underline{Y}_0(\underline{X}) + o(1) \Rightarrow$

$$\underline{Y}_0' = 0 \qquad\qquad \underline{Y}_0(0) = 0$$

whose soln $\underline{Y}_0(\underline{X}) = 0$ can't be matched to outer $y_0(x) = e^{-x}$!!