Matching (Theory)

To complete the inner approximation one must "match" it to the outer soln

$$(1) \qquad \lim_{x \to 0^+} y_0(x) = \lim_{X \to +\infty} Y_0(X)$$

An explanation of why this must be true (in most instances) is given here.

The outer expansion presumes $x > 0$ is fixed as $\varepsilon \to 0^+$

$$y(x, \varepsilon) = y_0(x) + o(1) \qquad x\text{-fixed}$$

so that

$$(2) \qquad \lim_{\substack{\varepsilon \to 0^+ \\ x\text{-fixed}}} \left(y(x,\varepsilon) - y_0(x) \right) = 0$$

Similarly since $x = \delta X$, $0 < \delta \ll 1$ and

$$y(x, \varepsilon) = Y(X, \varepsilon) = Y_0(X) + o(1) \qquad X \text{ fixed}$$

we have

$$(3) \qquad \lim_{\substack{\varepsilon \to 0^+ \\ X\text{-fixed}}} \left(y(\delta X, \varepsilon) - Y_0(X) \right) = 0$$

Each (2)-(3) presumes x, X are fixed.

We try to extend the validity of these limits using an "intermediate" variable z.

Intermediate variable

$$z = \frac{x}{\eta(\varepsilon)} \qquad \delta(\varepsilon) \ll \eta(\varepsilon) \ll 1$$

Since $\delta \ll \eta$, $z = O(1)$ for a thicker layer of thickness $O(\eta)$.

If we assume (2)-(3) are also true for z-fixed we have

$$(4) \qquad \lim_{\varepsilon \to 0^+} \left(y(\eta z, \varepsilon) - y_0(\eta z) \right) = 0$$

$$(5) \qquad \lim_{\varepsilon \to 0^+} \left(y(\eta z, \varepsilon) - Y_0\left(\frac{\eta z}{\delta}\right) \right) = 0$$

where z is fixed in both!

If these limits exist then we can subtract them and $y(\eta z, \varepsilon)$ terms cancel

$$\lim_{\varepsilon \to 0^+} \left(y_0(\eta z) - Y_0\left(\frac{\eta z}{\delta}\right) \right) = 0$$

Since $\eta z \to 0$ (i.e. $\eta \ll 1$) for z fixed, and since $\frac{\eta z}{\delta} \to +\infty$ (i.e, $\delta \ll \eta$) for z fixed we arrive at

$$(6) \qquad \lim_{x \to 0} y_0(x) = \lim_{X \to \infty} Y_0(X)$$

which is the matching condition we've been using.

Some Remarks

a) The validity of (6) hinges on whether the statements (4)-(5) are true. Each state that the outer and inner approximations have what's called an "extended region of validty". Extension Thms exist but in practical settings matchin is considered on a case by case basis

For example (4) may only be true if more terms of the outer soln are included, as in

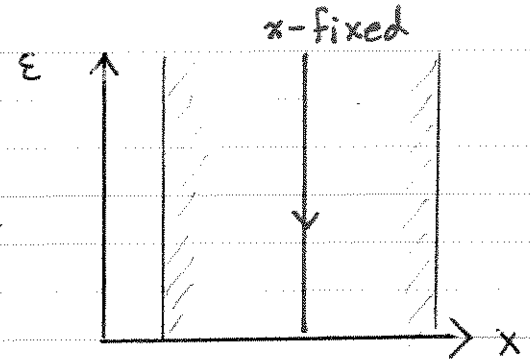
$$\lim_{\varepsilon \to 0^+} \left(y(\eta z, \varepsilon) - y_0(\eta z) - \varepsilon y_1(\eta z) \right) = 0$$

Further discussion is beyond the scope of this course but in general matching conditions look like

$$\lim_{\varepsilon \to 0^+} \frac{\sum_{K=0}^{P} \varepsilon^K y_K(\eta z) - \sum_{K=0}^{Q} \delta^K \mathrm{Y}_K\left(\frac{\eta z}{\delta}\right)}{\varepsilon^\alpha} = 0$$

which states a P-term outer soln matches a Q-term inner soln to $O(\varepsilon^\alpha)$

b) Fixed x, X, z limit processes can be represented graphically outer

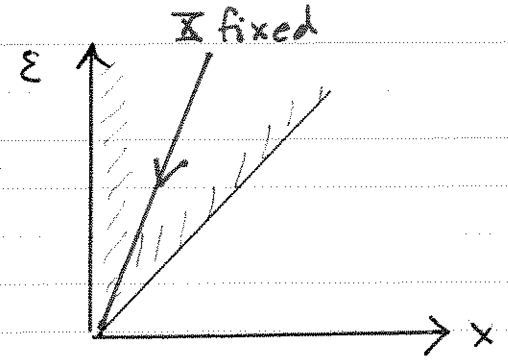


$x > 0$ fixed, $\varepsilon \to 0$

$y(x,\varepsilon) - y_0(x) = o(1)$

on this region

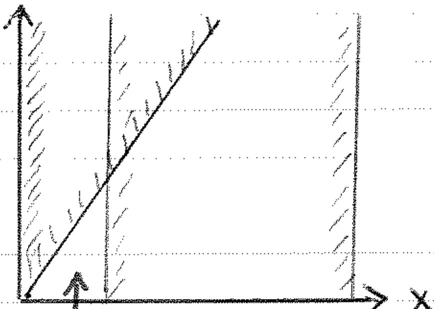
inner



$\bar{X} > 0$ fixed $(\delta(\varepsilon) = \varepsilon)$

$y(\delta \bar{X}, \varepsilon) - \bar{Y}_0(\bar{X}) = o(1)$

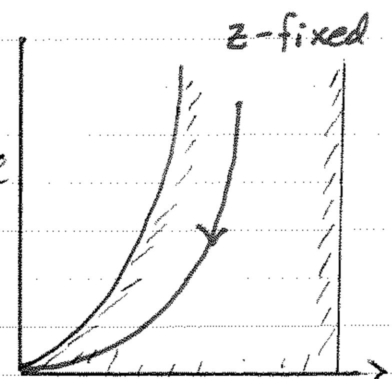
on this region
(Boundary layer $O(\varepsilon)$ thick)



domains of validy don't overlap.

Although $y_0(x)$ and $\bar{Y}_0(\bar{X})$ approximate the same function $y(x,\varepsilon)$ they don't (in the gap) unless their domains can be "extended"

intermediate



$y_0, \bar{Y}_0$ approximate y in gap now.

If $z = \frac{x}{\eta}$, $\varepsilon \ll \eta \ll 1$

then (say for $\eta = \sqrt{\varepsilon}$) fixed z, $\varepsilon \to 0$ limits extend the domain