Initial Layer Problems

In initial value problems initial (rather than boundary) conditions are stipulated.

For instance

(1) $$\varepsilon y'' = f(y, y', \varepsilon)$$

(2) $$y(0) = A$$

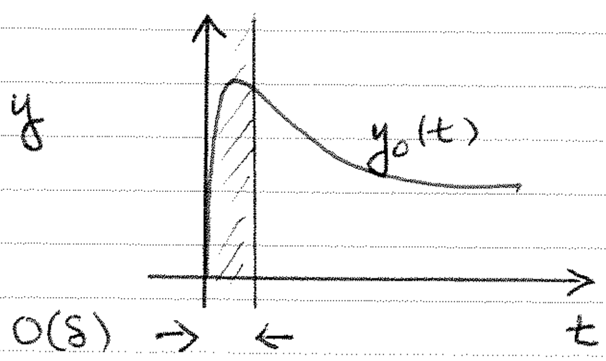
(3) $$y'(0) = B$$

is a class of such second order IVP. Here the $\varepsilon y''$ can be large if the derivative y' is large – making the problem singular.

Here an outer expansion $y(x,\varepsilon) = y_0(x) + o(1)$ yields the leading order problem

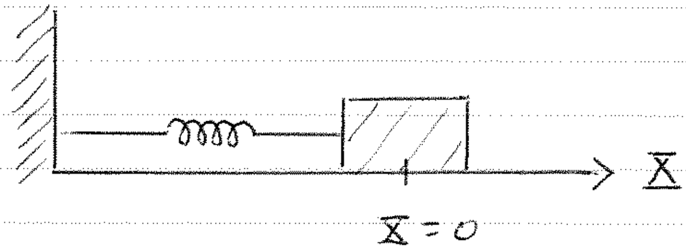
(4) $$0 = f(y_0, y_0', 0)$$

Since this is a first order diff eqn, its solution can't (in general) satisfy both the initial conditions (1)-(2).



Thus we expect initial (inner) layers.

Damped Spring Problem



Mass m at position $\bar{X}$ at time $\bar{T}$ acted on by two forces

$$F_{spring} = -k\bar{X} \qquad \text{(Hooke's Law)}$$

$$F_{friction} = -a\frac{d\bar{X}}{dt}$$

In dimensional form Newton's Laws $\Rightarrow$

$$(1) \qquad m\frac{d^2\bar{X}}{d\bar{T}^2} + a\frac{d\bar{X}}{d\bar{T}} + k\bar{X} = 0$$

Will assume initially mass is at rest at its equilibrium position $\bar{X} = 0$ and an impulse (momentum) of magnitude I is applied

$$(2) \qquad \bar{X}(0) = 0$$

$$(3) \qquad m\bar{X}'(0) = I$$

Nondimensionalize model

There are several nondimensionalizations (nonunique). We present one here

$$y = \frac{X}{X^*} \qquad t = \frac{T}{T^*}$$

Differential Eqn becomes

$$\frac{m}{k(T^*)^2}\frac{d^2y}{dt^2} + \frac{a}{kT^*}\frac{dy}{dt} + y = 0$$

↑ choose T^* to make this equal one

For $T^* = a/k$ we get

(4) $$\varepsilon y'' + y' + y = 0$$

where the dimensionless parameter ε is

(5) $$\varepsilon \equiv \frac{mk}{a^2}$$

Now we nondimensionalize initial conditions.

Independent of the choice of X^*

(6) $$y(0) = 0$$

The as yet to be defined constant X^* does enter in the other initial condition.

we have

$$\left(\frac{m \bar{X}^*}{T^*}\right) y'(0) = I$$

or for $T^* = a/k$, $\varepsilon = mk/a^2$

$$y'(0) = \frac{I a}{m k \bar{X}^*}$$

$$(7) \qquad \boxed{y'(0) = \frac{1}{\varepsilon}\left(\frac{I \bar{X}^*}{a}\right)}$$

we will choose $\bar{X}^*$ so that

$$(8) \qquad \varepsilon y'(0) = \alpha \qquad\qquad \alpha = \frac{I \bar{X}^*}{a}$$

where α is some dimensionless parameter. It's significance we will address later.

Model Problem

$$(9) \qquad \varepsilon y'' + y' + y = 0$$

$$(10) \qquad y(0) = 0$$

$$(11) \qquad \varepsilon y'(0) = \alpha$$

Has a known exact solution.

The exact solution is

$$y(x, \varepsilon) = c(\varepsilon)\left(e^{\lambda_+(\varepsilon) t} - e^{\lambda_-(\varepsilon) t} \right)$$

where

$$\lambda_\pm(\varepsilon) = \frac{1}{2\varepsilon}\left(-1 \pm \sqrt{1 - 4\varepsilon}\right)$$

$$c(\varepsilon) = \frac{\alpha}{\sqrt{1 - 4\varepsilon}}$$

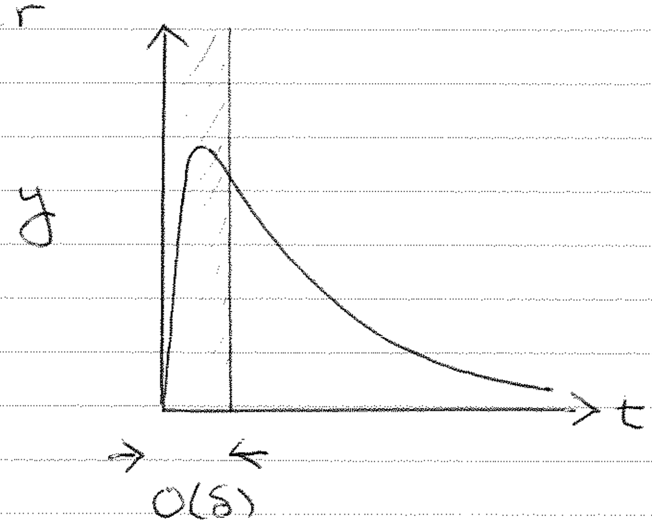
None of this nondimensionalization or exact solution depends on knowing the magnitudes of ε, α

Overdamped Spring Assumption

$$\varepsilon = \frac{mk}{a^2} \ll 1 \qquad \alpha = O(1)$$

where $\varepsilon \ll 1$ since friction coefficient a is large.

In this instance the exact solution has a layer



Matched asymptotic approximation

(1) $\varepsilon y'' + y' + y = 0 \quad , \quad t > 0$

(2) $y(0) = 0 \qquad \varepsilon y'(0) = \alpha$

where $0 < \varepsilon \ll 1$ and $\alpha = O(1)$

Outer solution $y(t, \varepsilon) = y_0(t) + o(1)$

$$y_0' + y_0 = 0$$

satisfies *no* intial conditions since it is an approximation valid only away from the layer at $t = 0$.

$$y_0(t) = A e^{-t}$$

will find constant A from matching condition.

Inner solution

$$y(t, \varepsilon) = Y(\tau, \varepsilon) \qquad \tau = \frac{t}{\delta(\varepsilon)} \quad , \quad \delta \ll 1$$

yields

$$\frac{\varepsilon}{\delta^2} Y'' + \frac{1}{\delta} Y' + Y = 0$$

$$① \sim ② \gg ③$$

Hence choose

$$\delta(\varepsilon) = \varepsilon$$

yields (exactly)

$$(3) \qquad \bar{Y}'' + \bar{Y}' + \varepsilon \bar{Y} = 0$$

More importantly the initial condit $\varepsilon y'(0) = \alpha$ becomes

$$(4) \qquad \varepsilon y'(0) = \bar{Y}'(0) = \alpha$$

Thus $\bar{Y}(\tau, \varepsilon) = \bar{Y}_0(\tau) + o(1)$ yields the leading order inner problem

$$(5) \qquad \bar{Y}_0'' + \bar{Y}_0' = 0$$

$$(6) \qquad \bar{Y}_0(0) = 0 \qquad \bar{Y}_0'(0) = \alpha$$

only if $\alpha = O(1)$. The inner soln $\bar{Y}_0(\tau)$ is

$$\bar{Y}_0(\tau) = \alpha(1 - e^{-\tau}) \qquad \tau = \frac{t}{\varepsilon}$$

Matching

Note $\tau \to +\infty$ away from layer

$$M = \lim_{\tau \to +\infty} \bar{Y}_0(\tau) = \lim_{t \to 0^+} y_0(t)$$

$$M = \alpha = A$$

Hence the outer soln is now complete

$$y_0(t) = \alpha e^{-t}$$

Uniform Approximation

$$y_u(t,\varepsilon) = y_0(t) + Y_0\left(\frac{t}{\varepsilon}\right) - M$$

$$y_u(t,\varepsilon) = \alpha\left(e^{-t} - e^{-t/\varepsilon}\right)$$

is a very simple result.

Issues Recall dimensionless parameters

$$\varepsilon = \frac{mk}{a^2} \qquad \alpha = \frac{I X^*}{a}$$

These dimensionless parameters can be independently small or large. For instance, if a is very large we expect $\varepsilon \ll 1$ but then why not $\alpha \ll 1$? Even if a is large the dimensional quantity I may large be as well making $\alpha \simeq 1$ for some $X^* \simeq 1$.

So even if $\varepsilon \ll 1$ we have cases of α small, α "order one" and α large.

Turns out α merely determines the magnitude of the layer. Why?

Set $\qquad z(t) = \frac{1}{\alpha} y(t)$

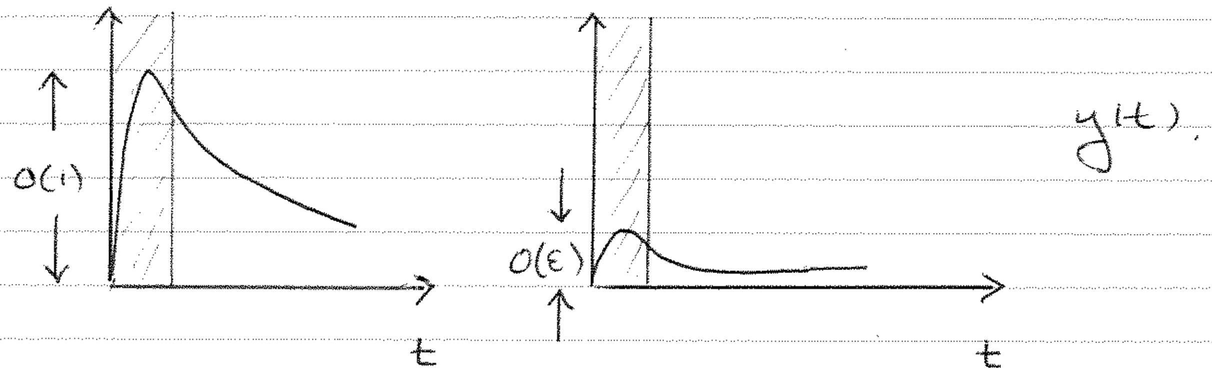
then

(1) $\varepsilon z'' + z' + z = 0$

(2) $z(0) = 0$

(3) $\varepsilon z'(0) = 1$

which has the same uniformly valid soln as if $\alpha = 1$



$\alpha = O(1)$ — large impulse I

$\alpha = O(\varepsilon)$ — small impulse I

Second Issue

The way one nondimensionalizes can affect how you solve a problem. Suppose as before we chose

$$T^* = a/K$$

Then the dimensionless differential equation is still

$$\varepsilon y'' + y' + y = 0$$

with the same ε. And, $y(0) = 0$ still.
One could have written $\bar{X}'(0) = I/m$ as

$$\left(\frac{m\bar{X}^*}{T^*}\right) y'(0) = I$$

$$y'(0) = \frac{Ia}{mk\bar{X}^*}$$

then chosen $\bar{X}^* = (Ia)/(mk)$ so that $y'(0) = 1$.
For this different scaling

(1) $$\varepsilon y'' + y' + y = 0$$

(2) $$y(0) = 0 \qquad y'(0) = 1$$

The inner problem for this new problem is different

$$y(t, \varepsilon) = \bar{Y}(\tau, \varepsilon) \qquad \tau = \frac{t}{\delta(\varepsilon)}$$

as before yields (for $\delta = \varepsilon$)

(3) $$\bar{Y}'' + \bar{Y}' + \varepsilon \bar{Y} = 0$$

(4) $$\bar{Y}(0) = 0 \qquad \frac{1}{\varepsilon}\bar{Y}'(0) = 1$$

If we now expand $\underline{Y}(\tau, \varepsilon)$:

$$\underline{Y}(\tau, \varepsilon) = \underline{Y}_0(\tau) + \varepsilon \underline{Y}_1(\tau) + O(\varepsilon^2)$$

we find

$O(1)$ $\quad \underline{Y}_0'' + \underline{Y}_0' = 0 \quad , \quad \underline{Y}_0(0) = \underline{Y}_0'(0) = 0$

$O(\varepsilon)$ $\quad \underline{Y}_1'' + \underline{Y}_1' = -\underline{Y}_0 \quad , \quad \underline{Y}_1(0) = 0, \ \underline{Y}_1'(0) = 1$

where, in particular, the initial conditions we derived from

$$\frac{1}{\varepsilon} \underline{Y}'(0, \varepsilon) = \frac{1}{\varepsilon} \left(\underline{Y}_0'(0) + \varepsilon \underline{Y}_1'(0) + O(\varepsilon^2) \right)$$

$$= \frac{1}{\varepsilon} \underline{Y}_0'(0) + \underline{Y}_1'(0) + O(\varepsilon)$$

$$= 1$$

So that $\underline{Y}_0'(0) = 0$, $\underline{Y}_1'(0) = 1$ etc.

Solution of $O(1)$ problem is $\underline{Y}_0(\tau) \equiv 0$ thus

$$\underline{Y}_1'' + \underline{Y}_1' = 0 \qquad \underline{Y}_1(0) = 0 \quad \underline{Y}_1'(0) = 1$$

and $\underline{Y}_1(\tau) = 1 - e^{-\tau}$. Then $\underline{Y} = \varepsilon \underline{Y}_1(\tau) + o(\varepsilon)$ must be matched to a scaled outer soln $y(t, \varepsilon) = \varepsilon y_0(t) + o(1)$.

Point is, everything is nastier if one makes a bad choice for nondimensionalization.