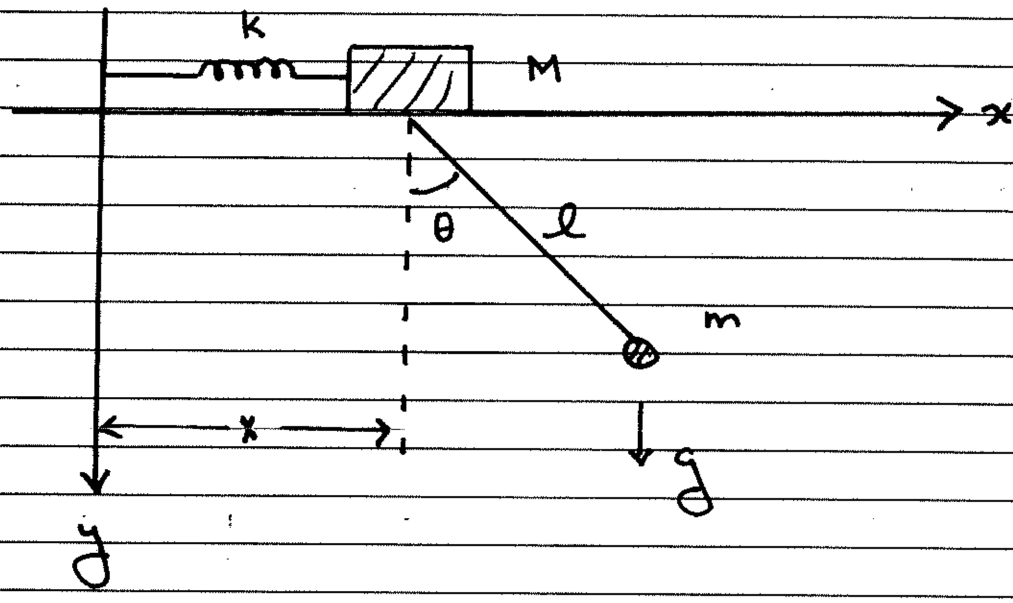
EXAMPLE Spring Pendulum system



Mass M slides on x axis with no friction attached to wall by spring of stiffness k.

Mass m attached to M swings on uniplanar firictionless pendulum of length l.

Position vectors

$$(x, y) = (x, 0)$$

$$(X, Y) = (x + l\sin\theta, l\cos\theta)$$

of mass M and m, respectively.

Potential Energies

$$U_1 = \tfrac{1}{2} k x^2 \qquad \text{mass } M \text{ (spring)}$$

$$U_2 = -mgl\cos\theta \qquad \text{mass } m \text{ (gravity)}$$

Kinetic Energies

The kinetic energy of mass M is easy:

$$T_1 = \tfrac{1}{2} M \dot{x}^2$$

For the smaller mass m

$$T_2 = \tfrac{1}{2} m \left(\dot{X}^2 + \dot{Y}^2 \right)$$

$$T_2 = \tfrac{1}{2} m \left\{ (\dot{x} + l\cos\theta\,\dot{\theta})^2 + (l\sin\theta\,\dot{\theta})^2 \right\}$$

$$T_2 = \tfrac{1}{2} m \left\{ \dot{x}^2 + 2\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2 \right\}$$

Lagrangian

$$L(x, \dot{x}, \theta, \dot{\theta}) = \underbrace{T_1 + T_2}_{\text{total kinetic}} - \underbrace{(U_1 + U_2)}_{\text{total potential}}$$

Principle of Least Action in physics states that the motion in the system is that which extremizes the "Action"

$$J \equiv \int_{t_0}^{t_1} L(x, \dot{x}, \theta, \dot{\theta})\, dt$$

Thus the equations of motion are given by the Euler-Lagrange equations

$$(1) \qquad L_x = \frac{d}{dt} L_{\dot{x}}$$

$$(2) \qquad L_\theta = \frac{d}{dt} L_{\dot{\theta}}$$

Equations (1)-(2) are two nonlinear second order equations (fourth order system)

These are <u>necessary</u> for extrema.

Calculations reveal :

$$L_x = -kx$$

$$L_{\dot{x}} = \mu \dot{x} + m \cos\theta \, \dot{\theta} \qquad \mu \equiv m + M$$

$$L_\theta = -m \dot{x} \dot{\theta} \sin\theta - mg\ell \sin\theta$$

$$L_{\dot{\theta}} = m \dot{x} \cos\theta + m\ell^2 \dot{\theta}$$

Explicitly EL-eqns (1)-(2) are

$$-kx = \frac{d}{dt}\left(\mu \dot{x} + m \cos\theta \, \dot{\theta}\right)$$

$$-m \dot{x} \dot{\theta} \sin\theta - mg\ell \sin\theta = \frac{d}{dt}\left(m \dot{x} \cos\theta + m\ell^2 \dot{\theta}\right)$$

which are very complicated and hard to solve. Can be done numerically.

$$\boxed{\begin{aligned} \mu \ddot{x} + m\cos\theta \, \ddot{\theta} - m \sin\theta \, \dot{\theta}^2 + kx &= 0 \\ m\cos\theta \, \ddot{x} + m\ell^2 \ddot{\theta} + mg\ell \sin\theta &= 0 \end{aligned}}$$

Can also look for <u>first integrals</u> to reduce the order of the system.
No general rules/methods.