Linear Differential Equations (First Order)

$$a(x)\, y' + b(x)\, y = c(x) \qquad \text{nonhomogeneous}$$

$$a(x)\, y' + b(x)\, y = 0 \qquad \text{homogeneous}$$

$$(1) \qquad y' + p(x)\, y = q(x) \qquad \text{standard form.}$$

Solution of standard form found using an integrating factor $\mu(x)$:

$$(2) \qquad \mu y' + \mu p y = \mu q$$

Choosing

$$\mu(x) = \exp\left(\int^x p(t)\, dt\right)$$

the left side of (2) is an exact differential since $\mu' = \mu p$, i.e.,

$$\mu y' + \mu' y = \mu q$$

$$(3) \qquad (\mu y)' = \mu q$$

Integrating (3) one finds

$$y(x) = \frac{c}{\mu(x)} + \frac{1}{\mu(x)} \int^x \mu(t)\, q(t)\, dt$$

Sometimes written

$$y(x) = c\, y_h(x) + y_p(x)$$

where y_h and y_p are homogeneous and particular solutions of (1)

Differential operator notation

$$L[y] \equiv y' + p(x)\,y$$

then

$L[y_h] = 0$ homogenous solution

$L[y_p] = q(x)$ particular solution

Initial Value Problem (IVP)

The general solution of

(4) $$y' + p(x)\,y = q(x)$$

is not unique since $c \in \mathbb{R}$ is arbitrary. in

(5) $$y = c\,y_h(x) + y_p(x)$$

If $p(x)$ and $q(x)$ are continuous on some interval containing x_0 the the solution of the (IVP)

(6) $$y' + p(x)\,y = q(x)$$

(7) $$y(x_0) = y_0$$

is unique.

EXAMPLE $\boxed{y' + 2y = e^{-x}, \qquad y(0) = 3}$

$$p(x) = 2 \qquad q(x) = e^{-x}$$

Integrating factor

$$\mu(x) = \exp\left(\int^x 2\,dt\right) = e^{2x}$$

Particular solution

$$y_p(x) = \frac{1}{\mu(x)} \int^x \mu(t)\, q(t)\, dt = e^{-2x} \int^x e^t\, dt$$

$$y_p(x) = e^{-x}$$

Homogeneous solution $y_h(x) = \mu(x)^{-1} = e^{-2x}$

General solution

$$y(x) = c y_h(x) + y_p(x) \qquad c \in \mathbb{R}$$

$$y(x) = ce^{-2x} + e^{-x}$$

Use initial condition $y(0) = 3$ to find constant c

$$y(0) = c + 1 = 3 \qquad \Rightarrow \quad c = 2$$

Solution of IVP

$$y(x) = 2e^{-2x} + e^{-x}$$

EXAMPLE $\boxed{y' - 2xy = x \qquad y(0) = 1}$

$$p(x) = -2x \qquad q(x) = x$$

Integrating factor

$$\mu(x) = \exp\left(-\int^x 2t\,dt\right) = e^{-2x^2}$$

Particular solution

$$y_p(x) = \frac{1}{\mu(x)}\int^x \mu(t)q(t)\,dt = e^{2x^2}\int^x t e^{-2t^2}\,dt$$

$$y_p(x) = -\tfrac{1}{2} \qquad = e^{2x^2}\cdot\left(-\tfrac{1}{2}e^{-2x^2}\right)$$

Homogeneous solution $y_h(x) = \mu(x)^{-1} = e^{+2x^2}$

General solution

$$y(x) = c\,y_h(x) + y_p(x) \qquad c \in \mathbb{R}$$

$$y(x) = ce^{2x^2} - \tfrac{1}{2}$$

Use initial condition $y(0) = 1$ to find c

$$y(0) = c - \tfrac{1}{2} = 1 \qquad \Rightarrow \quad c = \tfrac{3}{2}$$

Solution of IVP

$$y(x) = \tfrac{3}{2}e^{2x^2} - \tfrac{1}{2}$$

EXAMPLE $\boxed{y' - \frac{1}{x} y = x \qquad y(2) = 8}$

$$p(x) = -\frac{1}{x}, \quad q(x) = x$$

Integrating factor

$$\mu(x) = \exp\left(-\int^x \frac{1}{t}\,dt\right) = \exp(-\ln x) = \frac{1}{x}$$

Particular solution

$$y_p(x) = \frac{1}{\mu(x)} \int^x \mu(t) q(t)\,dt = x \cdot \int^x \frac{1}{t} \cdot t\,dt$$

$$y_p(x) = x^2$$

Homogeneous solution $y_h(x) = \mu(x)^{-1} = x$

General Solution

$$y(x) = c\, y_h(x) + y_p(x)$$

$$y(x) = cx + x^2$$

Solution of IVP

$$y(2) = 2c + 4 = 8 \quad \Rightarrow \quad c = 2$$

$$y(x) = 2x + x^2$$

Note that in this case the "initial condition" was specified at $x_0 = 2$ not $x_0 = 0$.