# Separable Differential Equations

A first order differential equation is said to be separable if it can be written in the form

$$y' = f(x)\, g(y)$$

for some functions $f$ and $g$. Examples

$$y' = x y^2 \qquad f(x) = x \qquad g(y) = y^2$$

$$y' = x y \qquad f(x) = x \qquad g(y) = y$$

$$y' = \frac{x^2}{1+y^2} \qquad f(x) = x^2 \qquad g(y) = \frac{1}{1+y^2}$$

The general solution is found through the following integration

$$\frac{dy}{dx} = f(x)\, g(y)$$

$$\int \frac{dy}{g(y)} = \int f(x)\, dx$$

The solution of the initial value problem (IVP)

$$y' = f(x)\, g(y) \quad , \quad y(x_0) = y_0$$

is found from the definite integral

$$\int_{y_0}^{y} \frac{dz}{g(z)} = \int_{x_0}^{x} f(t)\, dt$$

EXAMPLE Find the general solution of $y' = xy^2$

$$\int \frac{dy}{y^2} = \int x\,dx$$

$$-\frac{1}{y} = \frac{1}{2}x^2 + c \qquad c \in \mathbb{R} \text{ constant}$$

$$y(x) = -\frac{2}{2c + x^2}$$

EXAMPLE A falling object with initial velocity $v_0$ and coefficient of friction $k$ has a velocity $v(t)$ which satisfies

$$\frac{dv}{dt} = g - kv \qquad v(0) = v_0$$

Find $v(t)$. Here $g = 32\text{ ft/sec}^2$ is grav. const.

$$\int_{v_0}^{v} \frac{dz}{g - kz} = \int_0^t dt = t$$

$$-\frac{1}{k}\log(g - kz)\Big|_{v_0}^{v} = t$$

$$\log\left(\frac{g - kv}{g - kv_0}\right) = -kt$$

$$v(t) = g/k - \left(g/k - v_0\right)e^{-kt}$$

Note $v(t) \to v_T = g/k$ (terminal velocity)

**EXAMPLE** Some solutions are <u>implicitly</u> defined

$$y' = \frac{x}{3y^2+1}$$

yields $\qquad y^3 + y = x + c$

<u>Orthogonal Curves and Separable Equations</u>

Suppose a family of curves are defined by the algebraic equation

$$(1) \qquad f(x, y) = c$$

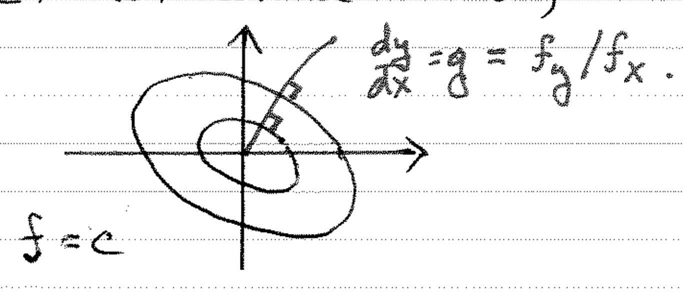
Using implicit differentiation one finds

$$(2) \qquad \frac{dy}{dx} = \frac{-f_x}{f_y}$$

Curves that are everywhere orthogonal ($\perp$) to those defined by (1) must have a slope equal to the negative reciprocal of that in equation (2):

$$(3) \qquad \frac{dy}{dx} = \frac{f_y}{f_x} = g(x,y)$$

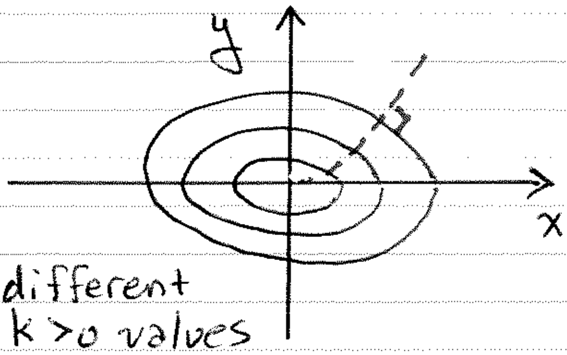
If (3) can be solved, orthogonal curves can be found.



**EXAMPLE** Let $a > 0$ and consider the family of curves (ellipses)

$$(1) \quad \frac{x^2}{a^2} + y^2 = k^2$$

$$f(x, y) = k^2$$




different $k > 0$ values

Show left are the curves and orthogonal curves (- - -)

The slope of the curves in (1) are found by implicit differentiation

$$\frac{2}{a^2} x + 2y \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{a^2 y} = m$$

Thus orthogonal curves have slope $-\frac{1}{m}$ and

$$\frac{dy}{dx} = \frac{a^2 y}{x}$$

which is separable and has solution

$$y(x) = c\, x^{a^2} \quad \Big\} \text{ orthog curves}$$

## Bernoulli Equation

$$\boxed{(1) \qquad y' + p(x)\, y = q(x)\, y^n}$$

can be transformed into a linear first order equation in

$$(2) \qquad u = y^{1-n}$$

In particular

$$\frac{1}{(1-n)} \frac{du}{dx} = y^{-n} \frac{dy}{dx}$$

Then noting (1) may be written

$$y^{-n} y' + p(x)\, y^{1-n} = q(x)$$

$$\frac{1}{(1-n)} u' + p(x)\, u = q(x)$$

$$(3) \qquad u' + (1-n) p(x)\, u = (1-n) q(x)$$

If one solves (3) for $u(x)$, $y(x)$ can be retrieved from (2).

<u>Historical Note</u>: Jacob Bernoulli (1690) showed the soln of a certain Bernoulli eqn solved the isochrone curve problem. An isochrone curve has a shape such that a mass falling along it under gravity (no fric) will end at its base in the same time independent of starting point.

EXAMPLE $\boxed{y' - \frac{2}{x} y = -x^2 y^2}$

Divide by $y^2$

$$y^{-2} y' - \frac{2}{x} y^{-1} = -x^2$$

then since $(y^{-1})' = y^{-2} y'$ we have

$$-u' - \frac{2}{x} u = -x^2 \qquad u = y^{-1}$$

$$\boxed{(1) \qquad u' + \frac{2}{x} u = +x^2 \qquad u = y^{-1}}$$

Integrating factor for the linear eqn (1) is

$$\mu(x) = \exp\left(\int^x \frac{2}{x} dx\right) = \exp(2 \ln x) = x^2$$

Multiply through by $\mu(x) = x^2$ yields

$$(x^2 u)' = x^4$$

$$x^2 u = \frac{1}{5} x^5 + c$$

$$u = \frac{1}{5} x^3 + \frac{c}{x^2}$$

$$\frac{1}{y} = \frac{1}{5} x^3 + \frac{c}{x^2}$$

Thus

$$y(x) = \frac{x^2}{\frac{1}{5} x^3 + c}$$

is the general solution.