Second order constant coefficient homogeneous

Let $a, b, c \in \mathbb{R}$ be constants and define the second order differential operator

$$L[y] = ay'' + by' + cy$$

We seek two (linearly independent) solutions of the homogeneous equation

(1) $L[y] = 0$

(When the right side does not equal zero it is non-homogeneous.)

To find these solutions we assume

(2) $y = e^{\lambda x}$

then $L[y] = 0$ implies

(3) $P(\lambda) = a\lambda^2 + b\lambda + c = 0$

Equation (3) is called the characteristic equation for the differential equation (1).

There are three cases to consider

$P(\lambda)$ has two real distinct roots $\lambda_1 \neq \lambda_2$

$P(\lambda)$ has one real repeated root $\lambda_1 = \lambda_2$

$P(\lambda)$ has a complex root $\lambda_1 = \alpha + i\beta$

where $i^2 = -1$. 
Without proof we summarize general solns

\[ \lambda_1 \neq \lambda_2 \text{ real} \quad y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} \]
\[ \lambda_1 = \lambda_2 \text{ real} \quad y = c_1 x e^{\lambda_1 x} + c_2 x e^{\lambda_2 x} \]
\[ \lambda_1 = \alpha + i\beta \quad y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x \]

**EXAMPLE**

\[
\begin{align*}
D'' + 5D' + 6D &= 0 \\
y(0) &= 0 \\
y'(0) &= 1
\end{align*}
\]

Initial Value Problem

**Characteristic equation**

\[ P = \lambda^2 + 5\lambda + 6 = (\lambda + 3)(\lambda + 2) = 0 \]

Thus \( \lambda_1 = -3 \), \( \lambda_2 = -2 \) and the general solution is

\[ y(x) = c_1 e^{-3x} + c_2 e^{-2x} \]

Use initial conditions to find \( c_1 \) and \( c_2 \).

\[ y'(x) = -3c_1 e^{-3x} - 2c_2 e^{-2x} \]

Evaluating \( y(x) \) and \( y'(x) \) at \( x = 0 \).

\[
\begin{align*}
y(0) &= c_1 + c_2 = 0 \\
y'(0) &= -3c_1 - 2c_2 = 1
\end{align*}
\]

From which \( c_1 = 1 \) and \( c_2 = -1 \) so

\[ y(x) = e^{-2x} - e^{-3x} \]
**EXAMPLE**  
Find the general soln of \( y'' + 4y' + 4y = 0 \)

Characteristic equation
\[
P = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2 = 0
\]

Thus \( \lambda_1 = \lambda_2 = -2 \) and
\[
y(x) = c_1 e^{-2x} + c_2 x e^{-2x}
\]

**EXAMPLE**  
Solve the initial-value problem
\[
y'' + 4y = 0, \quad y(0) = 0, \quad y'(0) = 1
\]

Characteristic Eqn
\[
P = \lambda^2 + 4 = 0 \quad \lambda = \pm 2i
\]

hence the general soln \((x = 0, \beta = 2)\) is
\[
y(x) = c_1 \cos(2x) + c_2 \sin(2x)
\]
\[
y'(x) = -2c_1 \sin(2x) + 2c_2 \cos(2x)
\]

Then
\[
y(0) = c_1 = 0
\]
\[
y'(0) = 2c_2 = 1
\]

implies \( c_1 = 0, \ c_2 = \frac{1}{2} \) and
\[
y(x) = \frac{1}{2} \sin(2x)
\]
Variation of Parameters (and order)

Seek a particular solution $y_p(x)$ of

$$L[y] = y'' + p(x)y' + q(x)y = f(x)$$

given we know two linearly independent solutions of the homogeneous problem

$$L[y_i] = 0 \quad i = 1, 2$$

In particular, we seek a particular solution of the form

$$y_p(x) = a(x)y_1(x) + b(x)y_2(x)$$

where the unknown functions $a(x), b(x)$ must be chosen so that

$$L[y_p] = f(x)$$

Eqn (2) is one equation for two unknown functions. We need another if we are to find $a, b$. The second is chosen in a "smart" way

$$a'y_1 + b'y_2 = 0$$

so that

$$y'_p = ay'_1 + by'_2$$

$$y''_p = ay''_1 + by''_2 + a'y_1' + b'y_2'$$
Using these expressions in $L[y_p] = f$ yields

$$L[y_p] = aL[y_1] + bL[y_2] + a'y_1 + b'y_2$$

The first terms cancel since $L[y_k] = 0$, hence

$$(4) \quad a'y_1 + b'y_2 = f(x)$$

Conclude $y_p(x)$ is a particular soln if both (3) and (4) are satisfied

$$(5) \quad a'y_1 + b'y_2 = 0$$
$$a'y_1' + b'y_2' = f$$

In matrix notation

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

Solve for unknowns $a', b'$

$$(6) \quad \begin{bmatrix} a' \\ b' \end{bmatrix} = \frac{1}{W(x)} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ f \end{bmatrix}$$

where $W(x) = y_1y_2' - y_1'y_2$ is the Wronskian of $y_1$ and $y_2$. Thus $y_1'y_2$ is the Wronskian

$$(7) \quad a(x) = -\frac{1}{W(x)} y_2(x) f(x)$$

$$(8) \quad b'(x) = +\frac{1}{W(x)} y_1(x) f(x)$$

which can be integrated in $x$ to get $a(x), b(x)$.
Having found $a(x)$, $b(x)$ from (9)-(10), we have found

\begin{equation}
(11) \quad y_p(x) = a(x) y_1(x) + b(x) y_2(x)
\end{equation}

Remarks

(i) To invert the matrix system to derive eqn (6) we used the fact that for any invertible $A \in \mathbb{R}^{2 \times 2}$

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}
\]

where the determinant $\det A = ad - bc$.

(ii) Equations (9)-(11) are often written as the sole integral

\[y_p = \int_{x}^{x} \frac{y_2(x)(y_1(t) - y_1(x) y_2(t))}{-W(t)} f(t) \, dt\]

where, again

\[-W(t) = y_1(t) y_2'(t) - y_2(t) y_1'(t)\]
EXAMPLE

Find the general solution of \( y'' + y = \sec x \).

Here \( f(x) = \sec x \) and \( L[y] = y'' + y \).

Homogeneous solutions

\[ y_1(x) = \cos x \quad y_2(x) = \sin x \]

yields \( W = y_1y_2' - y_1'y_2 = \cos^2 x + \sin^2 x = 1 \).

Thus

\[ a(x) = -\int \frac{1}{W(t)} y_2(t)f(t)dt = -\int \frac{\sin t}{\cos t} dt = \ln(\cos x) \]

\[ b(x) = +\int \frac{1}{W(t)} y_1(t)f(t)dt = +\int 1 dt = x \]

so that a particular solution is

\[ y_p(x) = a y_1 + b y_2 = \cos x \ln(\cos x) + x \sin x \]

The general solution is then

\[ y(x) = c_1 \cos x + c_2 \sin x + x \sin x + \cos x \ln(\cos x) \]

Remark

It is easily verified \( L[u + v] = L[u] + L[v] \) thus if \( y_1 \) is any homogeneous soln

\[ L[y_1 + y_2] = L[y_1] + L[y_2] = f \]

i.e. the general soln is the sum of \( y_2 \) and the general homogeneous soln.

\[ y_1 + y_2 = c_1 y_1 + c_2 y_2 \]
Method of Undetermined Coefficients

Useful for finding particular solutions \( y_p(x) \) of constant coefficient second order differential equations

\[
L[y] = ay'' + by' + cy = f(x)
\]

If \( f(x) \) has special forms one guesses a form for \( y_p(x) \) which depends on unknown coefficients \( A_1 \ldots A_n \) and then chooses (if possible) the \( A_k \) so that (1) is satisfied.

Here we present examples where \( f(x) \) is not a soln of the homogeneous problem \( L[y] = 0 \).

<table>
<thead>
<tr>
<th>( f(x) )</th>
<th>( y_p(x) )</th>
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<tbody>
<tr>
<td>( ae^{bx} )</td>
<td>( A_1 e^{bx} )</td>
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<tr>
<td>( \alpha \sin bx )</td>
<td>( A_1 \sin bx )</td>
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<tr>
<td>( \alpha \cos bx )</td>
<td>( A_2 \cos bx )</td>
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<tr>
<td>( p_n(x) = q_0 + q_1 x + \ldots + q_n x^n )</td>
<td>( A_0 + A_1 x + \ldots + A_n x^n )</td>
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<td>( (A_0 + A_1 x + \ldots + A_n x^n) e^{bx} )</td>
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</tbody>
</table>
EXAMPLE \[ L[y] = y'' + 3y' - y = e^x \]

Guess \( y_p(x) = A e^x \) then
\[ L[y_p] = 3A e^x = e^x \]
only if \( A = \frac{1}{3} \). Conclude \( y_p(x) = \frac{1}{3} e^x \)

EXAMPLE \[ L[y] = y'' + 2y' + y = \sin x + \cos x \]

Guess \( y_p(x) = A \sin x + B \cos x \) then
\[ L[y_p] = 2A \cos x - 2B \sin x = \sin x + \cos x \]
only if \( 2A = 1 \), \(-2B = 1 \). Conclude \( y_p = \frac{1}{2} \sin x - \frac{1}{2} \cos x \)

EXAMPLE \[ L[y] = y'' - y' + y = x^2 - x \]

Guess \( y_p(x) = A x^2 + B x + C \)
\[ L[y] = A x^2 + (B - 2A)x + (2A + C - B) \]
\[ L[y] = x^2 - x \]

Conclude
\[
\begin{align*}
A &= 1 \\
B - 2A &= -1 \\
2A + C - B &= 0
\end{align*}
\]

\[
\begin{align*}
A &= 1 \\
B &= 1 \\
C &= -1
\end{align*}
\]

hence
\[ y_p(x) = x^2 + x - 1 \]
Initial Value Problems (2nd order)

If $p(x), q(x)$ are continuous on an interval containing $x_0$ then the following initial value problem has a unique solution:

1. $\{y\} = y'' + p(x)y' + q(x)y = f(x)
2. $y(x_0) = a$
3. $y'(x_0) = b$

**Example** Find the unique solution of

$$y'' - 3y' + 2y = 8x \quad y(0) = 4 \quad y'(0) = 2$$

Characteristic eqn $\lambda^2 - 3\lambda + 2 = 0$ has roots $\lambda = 1, 2$. Homogeneous soln $y_h(x) = c_1 e^x + c_2 e^{2x}$.

Use undetermined coefficients with $y_p(x) = Ax + B$ to find $y_p(x) = 6 + 4x$ so general soln is

$$y(x) = 6 + 4x + c_1 e^x + c_2 e^{2x}$$

$y'(x) = 4 + c_1 e^x + 2c_2 e^{2x}$

Use initial conditions:

$$\begin{align*}
y(0) &= 6 + c_1 + c_2 = 4 \\
y'(0) &= 4 + c_1 + 2c_2 = 2
\end{align*}$$

Solving this system $c_1 = -2, c_2 = 0$ yields

$$y(x) = 6 + 4x - 2e^x$$