## Second order constant coefficient homogeneous

Let $a, b, c \in \mathbb{R}$ be constants and define the second order differential operator

$$L[y] = ay'' + by' + cy$$

We seek two (linearly independent) solutions of the homogeneous equation

$$(1) \quad L[y] = 0$$

(when the right side does not equal zero it is nonhomogeneous)

To find these solutions we assume

$$(2) \qquad y = e^{\lambda x}$$

then $L[y] = 0$ implies

$$(3) \quad P(\lambda) = a\lambda^2 + b\lambda + c = 0$$

Equation (3) is called the characteristic equation for the differential equation (1).

There are three cases to consider

$P(\lambda)$ has two real distinct roots $\lambda_1 \neq \lambda_2$

$P(\lambda)$ has one real repeated root $\lambda_1 = \lambda_2$

$P(\lambda)$ has a complex root $\lambda_1 = \alpha + i\beta$

where $i^2 = -1$.

Without proof we summarize general solns

$\lambda_1 \neq \lambda_2$ real $\quad y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

$\lambda_1 = \lambda_2$ real $\quad y = c_1 e^{\lambda_1 x} + c_2 x e^{\lambda_1 x}$

$\lambda_1 = \alpha \pm i\beta$ $\quad y = c_1 e^{\alpha x} \cos\beta x + c_2 e^{\alpha x} \sin\beta x$

**EXAMPLE**

$$\boxed{\begin{array}{ll} y'' + 5y' + 6y = 0 & \\ y(0) = 0 & y'(0) = 1 \end{array}}$$

Initial Value Problem

Characteristic equation

$$P = \lambda^2 + 5\lambda + 6 = (\lambda + 3)(\lambda + 2) = 0$$

Thus $\lambda_1 = -3$, $\lambda_2 = -2$ and the general solution is

$$y(x) = c_1 e^{-3x} + c_2 e^{-2x}$$

Use initial conditions to find $c_1$ and $c_2$.

$$y'(x) = -3c_1 e^{-3x} - 2c_2 e^{-2x}$$

Evaluating $y(x)$ and $y'(x)$ at $x = 0$.

$$\left.\begin{array}{ll} y(0) = c_1 + c_2 = 0 \\ y'(0) = -3c_1 - 2c_2 = 1 \end{array}\right\} \text{Solve linear sys for } c_1, c_2$$

From which $c_1 = 1$ and $c_2 = -1$ so

$$y(x) = e^{-2x} - e^{-3x}$$

**EXAMPLE**

$$\boxed{\text{Find the general soln of } y'' + 4y' + 4y = 0}$$

Characteristic equation

$$P = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2 = 0$$

Thus $\lambda_1 = \lambda_2 = -2$ and

$$y(x) = c_1 e^{-2x} + c_2 x e^{-2x}$$

**EXAMPLE**

$$\boxed{\begin{array}{l}\text{Solve the initial value problem} \\ y'' + 4y = 0, \quad y(0) = 0, \; y'(0) = 1\end{array}}$$

Characteristic Eqn

$$P = \lambda^2 + 4 = 0 \qquad \lambda = \pm 2i$$

hence the general soln ($\alpha = 0$, $\beta = 2$) is

$$y(x) = c_1 \cos(2x) + c_2 \sin(2x)$$

$$y'(x) = -2c_1 \sin(2x) + 2c_2 \cos(2x)$$

Then

$$y(0) = c_1 = 0$$

$$y'(0) = 2c_2 = 1$$

implies $c_1 = 0$, $c_2 = \frac{1}{2}$ and

$$y(x) = \frac{1}{2} \sin(2x)$$

# Variation of Parameters (2nd order)

Seek a particular solution $y_p(x)$ of

$$L[y] \equiv y'' + p(x)y' + q(x)y = f(x)$$

given we know two linearly independent solutions of the homogeneous problem

$$L[y_i] = 0 \qquad i = 1, 2$$

In particular, we seek a particular solution of the form

$$(1) \qquad y_p(x) = a(x)\, y_1(x) + b(x)\, y_2(x)$$

where the unknown functions $a(x), b(x)$ must be chosen so that

$$(2) \qquad L[y_p] = f(x)$$

Eqn (2) is one equation for two unknown functions. We need another if we are to find $a, b$. The second is chosen in a "smart" way

$$(3) \qquad a' y_1 + b' y_2 = 0$$

so that

$$y_p' = a y_1' + b y_2'$$

$$y_p'' = a y_1'' + b y_2'' + a' y_1' + b' y_2'$$

Using these expressions in $L[y_p] = f$ yields

$$L[y_p] = a L[\cancel{y_1}] + b L[\cancel{y_2}] + a' y_1' + b' y_2'$$

The first terms cancel since $L[y_k] = 0$, hence

$$(4) \qquad a' y_1' + b' y_2' = f(x)$$

Conclude $y_p(x)$ is a particular soln if both (3) & (4) are satisfied

$$(5) \qquad \begin{aligned} a' y_1 + b' y_2 &= 0 \\ a' y_1' + b' y_2' &= f \end{aligned}$$

In matrix notation

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} a' \\ b' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$

Solve for unknowns $a', b'$

$$(6) \qquad \begin{pmatrix} a' \\ b' \end{pmatrix} = \frac{1}{W(x)} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ f \end{bmatrix}$$

where $W(x) = y_1 y_2' - y_1' y_2$ is the <u>Wronskian</u> of $y_1$ and $y_2$. Thus

$$(7) \qquad a'(x) = -\frac{1}{W(x)} y_2(x) f(x)$$

$$(8) \qquad b'(x) = +\frac{1}{W(x)} y_1(x) f(x)$$

which can be integrated in $x$ to get $a(x)$, $b(x)$.

$$\boxed{\begin{aligned} &(9) \qquad a(x) = -\int^{x} \frac{1}{W(t)}\, y_2(t) f(t)\, dt \\ &(10) \qquad b(x) = \int^{x} \frac{1}{W(t)}\, y_1(t) f(t)\, dt \end{aligned}}$$

Having found $a(x)$, $b(x)$ from (9)-(10) we have found

$$(11) \qquad y_p(x) = a(x) y_1(x) + b(x) y_2(x)$$

## Remarks

(i) To invert the matrix system to derive eqn (6) we used the fact that for any invertible $A \in \mathbb{R}^{2\times 2}$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where the determinant $\det A = ad - bc$.

(ii) Equations (9)-(11) are often written as the sole integral

$$y_p = \int^{x} \frac{y_2(x) y_1(t) - y_1(x) y_2(t)}{W(t)}\, f(t)\, dt$$

where, again

$$W(t) = y_1(t) y_2'(t) - y_2(t) y_1'(t)$$

**EXAMPLE** Find the general solution of $y'' + y = \sec x$.

Here $f(x) = \sec x$ and $L[y] = y'' + y$.
Homogeneous solutions

$$y_1(x) = \cos x \qquad y_2(x) = \sin x$$

yields $W = y_1 y_2' - y_1' y_2 = \cos^2 x + \sin^2 x = 1$.

Thus

$$a(x) = -\int^x \frac{1}{W(t)} y_2(t) f(t)\,dt = -\int^x \frac{\sin t}{\cos t}\,dt = \ln(\cos x)$$

$$b(x) = +\int^x \frac{1}{W(t)} y_1(t) f(t)\,dt = +\int^x 1\,dt = x$$

so that a particular solution is

$$y_p(x) = a y_1 + b y_2 = \cos x \ln(\cos x) + x \sin x$$

The general solution is then

$$y(x) = c_1 \cos x + c_2 \sin x + x \sin x + \cos x \ln(\cos x)$$

**Remark** It is easily verified $L[u+v] = L[u] + L[v]$ thus if $y_H$ is any homogeneous soln $L[y_p + y_H] = L[y_p] + L[y_H] = f$, i.e. the general soln is the sum of $y_p$ and the general homogenous soln.

$$y_H = c_1 y_1 + c_2 y_2$$

# Method of Undetermine Coefficients

Useful for finding particular solutions $y_p(x)$ of constant coefficient second order differential equations

(1) $L[y] = ay'' + by' + cy = f(x)$

If $f(x)$ has special forms one guesses a form for $y_p(x)$ which depends on unknown coefficents $A_1 \ldots A_n$ and then chooses (if possible) the $A_k$ so that (1) is satisfied.

Here we present examples where $f(x)$ is not a soln of the homogeneous problem $L[y]=0$.

| $f(x)$ | $y_p(x)$ |
|---|---|
| $\alpha e^{\beta x}$ | $A_1 e^{\beta x}$ |
| $\alpha \sin \beta x$<br>$\alpha \cos \beta x$ | $A_1 \sin \beta x$<br>$A_2 \cos \beta x$ |
| $P_n(x) = a_0 + a_1 x + \cdots a_n x^n$ | $A_0 + A_1 x + \cdots A_n x^n$ |
| $P_n(x) e^{\beta x}$ | $(A_0 + A_1 x + \cdots A_n x^n) e^{\beta x}$ |

EXAMPLE $L[y] = y'' + 3y' - y = e^x$

Guess $y_p(x) = Ae^x$ then

$$L[y_p] = 3Ae^x = e^x$$

only if $A = 1/3$. Conclude $y_p(x) = \frac{1}{3}e^x$

EXAMPLE $L[y] = y'' + 2y' + y = \sin x + \cos x$

Guess $y_p(x) = A\sin x + B\cos x$ then

$$L[y_p] = 2A\cos x - 2B\sin x = \sin x + \cos x$$

only if $2A = 1, -2B = 1$. Conclude $y_p = \frac{1}{2}\sin x - \frac{1}{2}\cos x$

EXAMPLE $L[y] = y'' - y' + y = x^2 - x$

Guess $y_p(x) = Ax^2 + Bx + C$

$$L[y] = Ax^2 + (B-2A)x + (2A+C-B)$$

$$L[y] = x^2 - x$$

Conclude

$$\left.\begin{aligned} A &= 1 \\ B-2A &= -1 \\ 2A+C-B &= 0 \end{aligned}\right\} \quad \begin{aligned} A &= 1 \\ B &= 1 \\ C &= -1 \end{aligned}$$

hence

$$y_p(x) = x^2 + x - 1$$

# Initial Value Problems (2nd order)

If $p(x)$, $q(x)$ are continuous on an interval containing $x_0$ then the following initial value problem has a unique solution:

$$(1) \qquad L[y] = y'' + p(x)y' + q(x)y = f(x)$$

$$(2) \qquad y(x_0) = a$$

$$(3) \qquad y'(x_0) = b$$

EXAMPLE Find the unique solution of

$$y'' - 3y' + 2y = 8x \qquad y(0) = \hat{4} \qquad y'(0) = \hat{2}$$

Characteristic eqn $\lambda^2 - 3\lambda + 2 = 0$ has roots $\lambda = 1, 2$. Homogeneous soln $y_h(x) = c_1 e^x + c_2 e^{2x}$.

Use undetermined coefficients with $y_p(x) = Ax + B$ to find $y_p(x) = 6 + 4x$ so general soln is

$$y(x) = 6 + 4x + c_1 e^x + c_2 e^{2x}$$

$$y'(x) = 4 + c_1 e^x + 2c_2 e^{2x}$$

Use initial conditions

$$\left.\begin{aligned} y(0) &= 6 + c_1 + c_2 = \hat{4} \\ y'(0) &= 4 + c_1 + 2c_2 = \hat{2} \end{aligned}\right\} \text{solve system for } c_1, c_2$$

Solving this system $c_1 = -2$, $c_2 = 0$ yields

$$y(x) = 6 + 4x - 2e^x$$