Systems of differential equations

Let t be the independent variable.

(1) $x_1' = a_{11} x_1 + a_{12} x_2$

(2) $x_2' = a_{21} x_1 + a_{22} x_2$

Generally a_{ij} can depend on t but we restrict to the case where they are constant.

The system (1)-(2) can be written in the compact matrix form

(3) $$\frac{d\vec{x}}{dt} = A\vec{x}$$

where

$$\vec{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

The system in (3) is "homogeneous". A nonhomogeneous system has the form

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}(t) \qquad \vec{f}(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$$

EXAMPLE Write $y'' + 2y' - 3y = 0$ as a system

Letting $x_1 = y$, $x_2 = y' = x_1'$ then $x_2' = y'' \Rightarrow$

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= -2x_2 + 3x_1 \end{aligned} \iff \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Solutions using eigenvalues, eigenvectors

We seek the general solution of

$$(1) \qquad \frac{d\vec{x}}{dt} = A\vec{x} \qquad \vec{x} \in \mathbb{R}^2$$

Toward this end we assume

$$(2) \qquad \vec{x}(t) = e^{\lambda t}\vec{y}$$

for some λ and vector $\vec{y}$. Using (2) in (1)

$$\lambda e^{\lambda t}\vec{y} = e^{\lambda t} A\vec{y}$$

from which we conclude

$$(3) \qquad (A - \lambda I)\vec{y} = 0$$

where $I \in \mathbb{R}^{2\times 2}$ is the identity matrix.

Equation (3) has nonzero solns $\vec{y}$ only if the matrix $(A - \lambda I)$ is not invertible:

$$(4) \qquad P(\lambda) = \det(A - \lambda I) = 0$$

Equation (4) is a quadratic in λ hence has roots λ_1, λ_2 called the <u>eigenvalues</u> of A. For these λ equation (3) has nonzero $\vec{y}_1, \vec{y}_2$ called <u>eigenvectors</u>.

Having found $\lambda_1, \lambda_2, \vec{y}_1, \vec{y}_2$ one can construct the general solution of the system (1).

Stated without proof are three forms for the general solution $\vec{x}(t)$.

Real Distinct Eigenvalues $\lambda_1 \neq \lambda_2$

$$\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{y}_1 + c_2 e^{\lambda_2 t} \vec{y}_2$$

Complex Eigenvalues $\lambda = \alpha \pm i\beta$

Since the eigenvalues λ are complex, so are the eigenvectors

$$(A - \lambda I)\vec{y} = \vec{0} \qquad \vec{y} = \vec{a} + i\vec{b}$$

then

$$\vec{x}(t) = c_1 e^{\alpha t}(\vec{a} \cos\beta t - \vec{b} \sin\beta t) + c_2 e^{\alpha t}(\vec{a} \sin\beta t + \cos\beta t\, \vec{b})$$

Repeated Eigenvalue $\lambda_1 = \lambda_2 \in \mathbb{R}$

Let λ_1, $\vec{y}_1$ and $\vec{z}_1$ solve

$$(A - \lambda_1 I)\vec{y}_1 = 0$$
$$(A - \lambda_1 I)\vec{z}_1 = \vec{y}_1$$

then

$$\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{y}_1 + c_2 e^{\lambda_1 t}(t\vec{y}_1 + \vec{z}_1)$$

EXAMPLE

Solve the following IVP

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 8 & -11 \\ 6 & -9 \end{bmatrix} \vec{x} \qquad \vec{x}(0) = \begin{pmatrix} 12 \\ 7 \end{pmatrix}$$

Characteristic Polynomial

$$P(\lambda) = \det\begin{bmatrix} 8-\lambda & -11 \\ 6 & -9-\lambda \end{bmatrix} = (8-\lambda)(-9-\lambda) + 66$$

$$P(\lambda) = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2)$$

Eigenvalues are roots of P

$$\lambda_1 = -3 \qquad \lambda_2 = 2$$

Eigenvectors

$$A - \lambda_1 I = \begin{bmatrix} 11 & -11 \\ 6 & -6 \end{bmatrix} \qquad \vec{y}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A - \lambda_2 I = \begin{bmatrix} 6 & -11 \\ 6 & -11 \end{bmatrix} \qquad \vec{y}_2 = \begin{pmatrix} 11 \\ 6 \end{pmatrix}$$

General Soln

$$\vec{x}(t) = c_1 e^{-3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{2t}\begin{pmatrix} 11 \\ 6 \end{pmatrix}$$

Initial Condition

$$\vec{x}(0) = \begin{pmatrix} c_1 + 11c_2 \\ c_1 + 6c_2 \end{pmatrix} = \begin{pmatrix} 12 \\ 7 \end{pmatrix}$$

Solve for c_1 and c_2 to conclude $c_1 = c_2 = 1$ and

$$\vec{x}(t) = \begin{pmatrix} e^{-3t} + 11e^{2t} \\ e^{-3t} + 6e^{2t} \end{pmatrix}$$

EXAMPLE

$$\boxed{\frac{d\vec{x}}{dt} = \begin{bmatrix} -2 & -3 \\ 3 & -2 \end{bmatrix} \vec{x} \qquad \text{general Solution}}$$

Characteristic Polynomial

$$P = \det \begin{bmatrix} -2-\lambda & -3 \\ 3 & -2-\lambda \end{bmatrix} = (\lambda+2)^2 + 9$$

Eigenvalues are roots: $(\lambda+2)^2 = -9$

$$\lambda = \alpha \pm i\beta = -2 \pm 3i \qquad \text{complex}$$

Complex eigenvector for $\lambda = -2+3i$

$$A - \lambda I = \begin{bmatrix} -2-(-2+3i) & -3 \\ 3 & -2-(-2+3i) \end{bmatrix} = \begin{bmatrix} -3i & -3 \\ 3 & -3i \end{bmatrix} \leftarrow$$

From the top row and $\vec{y} = (y_1, y_2)^T$ we need to choose any y_k such that

$$-3i\, y_1 - 3 y_2 = 0$$

Choose $y_2 = i$ and $y_1 = -1$

$$\underset{\alpha}{\lambda = -2} + \underset{\beta}{3}i \qquad \vec{y} = \begin{pmatrix} -1 \\ i \end{pmatrix} = \underbrace{\begin{pmatrix} -1 \\ 0 \end{pmatrix}}_{\vec{a}} + i \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{b}}$$

From the general result stated earlier

$$\vec{x}(t) = c_1 e^{-2t}\left(\cos 3t \begin{pmatrix} -1 \\ 0 \end{pmatrix} - \sin 3t \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$
$$+ c_2 e^{-2t}\left(\sin 3t \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \cos 3t \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$$

$$\vec{x}(t) = e^{-2t}\begin{pmatrix} -c_1 \cos 3t - c_2 \sin 3t \\ -c_1 \sin 3t + c_2 \cos 3t \end{pmatrix}$$

EXAMPLE

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \vec{x} \qquad \text{general solution}$$

Characteristic Polynomial

$$P(\lambda) = \det(A - \lambda I) = (2+\lambda)^2$$

Repeated eigenvalue $\lambda_1 = \lambda_2 = -2$. Thus there is only one eigenvector $\vec{y}_1$

$$(A - \lambda_1 I)\vec{y}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{y}_1 \qquad \vec{y}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

To complete general soln we need a "generalized" eigenvector $\vec{z}_1$ which is a solution to

$$(A - \lambda_1 I)\vec{z}_1 = \vec{y}_1$$

$$(1) \qquad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{z}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \vec{z}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Note that (like eigenvectors) there are many solns to (1). Any choice may be used for the general solution.

From previously stated result given $\vec{y}_1$, λ_1 and $\vec{z}_1$

$$\vec{x}(t) = c_1 e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{-2t} \left(t \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$$\vec{x}(t) = e^{-2t} \begin{pmatrix} c_1 + t c_2 \\ c_2 \end{pmatrix}$$

Fundamental Matrix Solns

$$(1) \qquad \frac{d\vec{x}}{dt} = A\vec{x}$$

Let $\vec{x}_1(t)$ and $\vec{x}_2(t)$ be two linearly independent solutions to (1). The general solution of (1) is then

$$(2) \qquad \vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t)$$

The fundamental matrix soln is the matrix having $\vec{x}_k(t)$ as columns

$$\underline{X}(t) = \left[\begin{array}{c|c} \vec{x}_1 & \vec{x}_2 \end{array} \right]$$

Such a matrix satisfies the matrix differential equation $\underline{X}' = A\underline{X}$. Also, the general solution has the compact form

$$\vec{x}(t) = \underline{X}(t)\vec{c} \qquad \vec{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Then, given some initial condition $\vec{x}_0$ we have

$$\vec{x}_0 = \underline{X}(0)\vec{c} \quad \Rightarrow \quad \vec{c} = \underline{X}(0)^{-1}\vec{x}_0$$

or

$$\vec{x}(t) = \underline{X}(t)\,\underline{X}(0)^{-1}\,\vec{x}_0$$

Fundamental matrices are generally used more for theory but have uses for finding particular solns.

Particular Solutions for systems

Let $\underline{X}(t)$ be a fundamental matrix for

$$(1) \qquad \frac{d\vec{x}}{dt} = A\vec{x}$$

we seek a particular solution $\vec{x}_p(t)$ of

$$(2) \qquad \frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}(t) \qquad \vec{f} = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}$$

Assume $\vec{x}_p(t) = \underline{X}(t)\,\vec{u}(t)$ for some as yet unknown $\vec{u}(t)$. Then

$$\frac{d\vec{x}_p}{dt} = \underline{X}'(t)\,\vec{u}(t) + \underline{X}(t)\frac{d\vec{u}}{dt}$$

but $\underline{X}' = A\underline{X}$ so (2) becomes

$$\cancel{A\underline{X}\vec{u}} + \underline{X}\vec{u}\,' = \cancel{A\underline{X}\vec{u}} + \vec{f}$$

$$\underline{X}\vec{u}\,' = \vec{f}$$

$$\vec{u}\,' = \underline{X}(t)^{-1}\vec{f}(t)$$

which we integrate in t to find $\vec{u}(t)$

Again recalling $\vec{x}_p = \underline{X}\vec{u}$ we conclude

$$(3) \qquad \vec{x}_p(t) = \underline{X}(t)\int^{t} \underline{X}(s)^{-1}\vec{f}(s)\,ds$$

So long as you can solve (1), Eqn (3) always yields $\vec{x}_p$ for the general soln

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \vec{x}_p(t)$$