Math 454 Final - Take Home (max 50 points)
Due: Friday, December 7, 2018 in class

Name: ______________________________

Instructions: You may use posted notes, your in-class notes and the textbook. You may not talk to anyone other than myself (and then only to clarify the meaning of the question). You may also use software such as Wolfram Alpha to plot functions or contours where appropriate. In the end, however, all work must be transcribed to this test hardcopy.

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1. [15] For the system

$$\dot{\mathbf{x}} = A\mathbf{x} = \begin{bmatrix} -3 & 2 \\ -4 & 3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad , \quad \mathbf{x}(0) = \mathbf{x}_0 \quad . \tag{1}$$

a) Sketch the phase portrait labelling $E^s(0)$ and $E^u(0)$.

b) Determine the flow function $\phi(t, \mathbf{x}_0)$ associated with (1)

2. [20] Consider the Hamiltonian system:

$$\begin{aligned} \dot{x} &= f_1(x,y) = 4y(1-y^2) \\ \dot{y} &= f_2(x,y) = 3(1-x^2)\ . \end{aligned}$$

a) Find the Hamiltonian $H(x,y)$.

b) Is the system reversible?

c) Draw the x and y nullclines indicating flow direction across them and the location of all equilibria. How many equilibria are there? (don't draw the entire phase portrait)

d) Is either of the fixed points $\bar{\mathbf{x}}_1 = (1, 1)$ and $\bar{\mathbf{x}}_2 = (1, 0)$ a nonlinear center? Why?

e) The trajectory through the origin corresponds to the $H(x, y) = 0$ level set. Solve $H(x, y) = 0$ for $y = y(x)$ noting $H(x, y) = 0$ is quadratic in $u = y^2$. Use these (four) solutions to plot the trajectory over the range $x \in [-2..1]$. You may use plotting software.

3. [5] By any means find the index of the equilibria at the origin where

$$\begin{aligned} \frac{dx}{dt} &= -x + y \\ \frac{dy}{dt} &= x + y \end{aligned}$$

4. [10] In each of the following statements a)-e) below, the system being referred to is a planar system

$$\dot{x} = f(x) \quad , \quad , \qquad x = (x_1, x_2)$$

where f is smooth on $\mathbb{R}^2$. State whether each statement is TRUE or FALSE.

a) A fixed point which is Liapunov stable (stable) must be attracting.

b) If a linear system has an isolated nonhyperbolic fixed point at $\bar{x} = (0, 0)$ it must be a center.

c) A reversible planar system has a heteroclinic connection from $\mathbf{x}_1 = (-1, 1)$ to $\mathbf{x}_2 = (1, 1)$. Such a system must have at least two heteroclinic connections.

d) Let $Df(x)$ be the Jacobian of $f(x)$. For every Hamiltonian system the trace of the Jacobian:

$$Tr(Df(x)) = 0 \qquad \forall x \in \mathbb{R}^2$$

e) It is possible for the x and y nullclines to be the same.