Math 454 Final - Take Home (max 50 points)
Due: Thursday, December 16, 2021 no later than 9:50am

Name: ______________________________

Instructions: You may use posted notes, your in-class notes and the textbook. You may not talk to anyone other than myself (and then only to clarify the meaning of the question). You may also use software such as Wolfram Alpha to plot functions or contours where appropriate.

You may

(a) turn it in early on Dec 15 (only) by sliding your named exam under my office door (Wil 2-236) or

(b) give it to me directly on Dec 16 between 8:00-9:50am in our regular classroom location where I will be.

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1. [10] In each of the following statements a)-e) below, the system being referred to is a planar system

$$\dot{x} = f(x) \quad , \qquad x = (x_1, x_2)$$

where f is smooth on $\mathbb{R}^2$. State whether each statement is TRUE or FALSE.

a) Saddles are hyperbolic fixed points

T **F**

b) Conservative systems are also Hamiltonian

T **F**

c) Some gradient systems are Hamiltonian

T **F**

d) Some Hamiltonian systems have an attracting fixed point

T **F**

e) If $\bar{x} = (0, 0)$ is an <u>isolated nonhyperbolic</u> of a <u>linear</u> system then it must be a center

T **F**

2. [15] Consider the reversible Hamiltonian system:

$$\begin{aligned} \dot{x} &= f_1(x,y) = y(1-y^2) \\ \dot{y} &= f_2(x,y) = 1-x^2 \ . \end{aligned}$$

a) Find the Hamiltonian $H(x,y)$.

b) Draw the x and y nullclines indicating flow direction across them and the location of all equilibria. How many equilibria are there? (don't draw the entire phase portrait)

c) Is either of the fixed points $\bar{x}_1 = (1,1)$ and $\bar{x}_2 = (1,0)$ a nonlinear center? Why?

3. [5] Show that the planar system defined by

$$\begin{aligned} \dot{x} &= f_1(x,y) = -y^2 + 3x^2 \\ \dot{y} &= f_2(x,y) = -2xy - 2y \end{aligned}$$

can have no periodic orbits by showing it is a gradient system. Specifically, find a potential function V such that $\mathbf{f} = (f_1, f_2) = -\nabla V$.

4. [15] For each of the following systems draw phase portraits for the cases $\mu < 0, \mu = 0, \mu > 0$ and then draw a bifurcation diagram of the x coordinate of the fixed point branches versus μ. You will need to compute the Jacobian for each fixed point branch in order to determine the fixed point stability.

$$\begin{aligned} \dot{x} &= \mu x - x^2 \quad \dot{y} = -y \\ \dot{x} &= \mu x + x^3 \quad \dot{y} = -y \end{aligned}$$

What types of bifurcations are these?

5. [5] The flow for a planar system is described by the following polar differential equations.

$$\begin{aligned} \dot{r} &= f(r) = r(1-r)(2-r) \\ \dot{\theta} &= g(\theta) \end{aligned}$$

where $g(\theta) > 0$. What are the limit cycles and their stability? Sketch the phase portrait.