Math 454 (2021)

Assignment 3 (Due: Thursday, October 14, 2021 in class)

1. [12pt] (SN) Consider the following differential equations where $x \in \mathbb{R}, \mu \in \mathbb{R}$:

$$\begin{aligned} \dot{x} &= f(x,\mu) = \mu - x(x-4)^2 & (1) \\ \dot{x} &= f(x,\mu) = \mu x - e^{\mu x} + \mu & (2) \end{aligned}$$

For (1)-(2)

i) identify all nonhyperbolic fixed points (μ^*, x^*) by solving $f = 0, f_x = 0$.

ii) which (if any) are saddle nodes using the theorem conditions: $f_\mu \neq 0, f_{xx} \neq 0$

iii) Find the second order Taylor series approximation of f in (x, μ) about each (μ^*, x^*)

iv) Find the "normal form" of all saddle nodes in (η, y) where $y = x - x^*, \eta = \mu - \mu^*$. This should look like $\dot{y} = \eta \pm ay^2 + O(3)$ where a possibly depends on the new parameter η.

2. [12pt] (TC) Consider the following differential equations where $x \in \mathbb{R}, \mu \in \mathbb{R}$:

$$\begin{aligned} \dot{x} &= f(x,\mu) = \mu x - ln(x+1) & (3) \\ \dot{x} &= f(x,\mu) = x(\mu - e^x) & (4) \end{aligned}$$

For (3)-(4)

i) identify all nonhyperbolic fixed points (μ^*, x^*) by solving $f = 0, f_x = 0$.

ii) determine which are transcritical bifurcations using the theorem conditions: $f_\mu = 0, f_{xx} \neq 0, f_{x\mu} \neq 0$

iii) Find the second order Taylor series approximation of f in (x, μ) about each (μ^*, x^*)

iv) Find the "normal form" for (3)-(4) of all transcritical bifurcations in (η, y) where $y = x - x^*, \eta = \mu - \mu^*$. This should look like $\dot{y} = ay(b \pm y) + O(3)$ where a, b possibly depend on the parameter η.

3. [6] Define the system

$$\dot{x} = f(x, \mu, \lambda) = \lambda x + \mu x^2 - x^3 = xg(x, \mu, \lambda) \tag{5}$$

where λ, μ are parameters. When $\lambda = 0$ the system has a pitchfork in μ. Find the locus of nonhyperbolic points Γ, i.e. the curve in the (μ, λ) plane defined by:

$$\begin{aligned} f(x,\mu,\lambda) &= 0 & (6) \\ f_x(x,\mu,\lambda) &= 0 & (7) \end{aligned}$$

Plot Γ and deduce (shade in) the region in the (μ, λ) plane where f has three roots - use calculus to prove your result and write out explicit formulae for all three real roots when they exist.