Homework 5 (Math 454) **Due: Thursday, December 2, 2021.**

---

1. [5] Find the flow function $\phi(t, x_0, y_0)$ for the following planar system

$$\begin{aligned} \dot{x} &= -x \quad , \quad x(0) = x_0 \\ \dot{y} &= x(1-y) \quad , \quad y(0) = y_0 \end{aligned}$$

If $x_0 \neq 0$, what does $\phi(t, x_0, y_0)$ approach as $t \to \infty$.

2. [5] Define the system

$$\begin{aligned} \dot{x} &= \alpha y^3 \\ \dot{y} &= \beta x^3 \end{aligned}$$

and function

$$E = \frac{x^4}{a^4} + \frac{y^4}{b^4}$$

a) Show E is a first integral if $a^4\beta + \alpha b^4 = 0$.

b) Compute the Jacobian of the system about the origin (sole fixed point). Does the linearized system indicate the system has a center?

c) Is the origin a hyperbolic fixed point?

d) Sketch the trajectories of the system when E is a first integral. Assume $\alpha > 0$ and $\beta > 0$ when determining the direction of the trajectories.

3. [10] For each of the following Hamiltonian systems, find the Hamiltonian $H(x, y)$. Use the level sets of H to draw an accurate phase portrait for the system indicating flow direction.

a) $$\begin{aligned} \dot{x} &= -x^3 \\ \dot{y} &= 3x^2 y \end{aligned}$$

b) $$\begin{aligned} \dot{x} &= 1 - y \\ \dot{y} &= x - 1 \end{aligned}$$

4. [10] Consider the system

$$\begin{aligned} \dot{x} &= y(1-x^2) \\ \dot{y} &= 1 - y^2 \end{aligned}$$

i) Is the system Hamiltonian?

ii) Is the system reversible?

iii) Find all the fixed points and deduce which (if any) of the fixed points are hyperbolic.

iv) Sketch the phase portrait (indicating flow direction).

5. [5] Convert the following system to polar coordinates:

$$\begin{aligned} \dot{x} &= -y + y(1 - x^2 - y^2) \\ \dot{y} &= x + x(1 - x^2 - y^2) \end{aligned}$$