M455: Assignment 8

Due: March, Monday 25, 2019.

1. [10] *Linear stability of fixed points of 1-D maps:*

Use linear stability analysis to categorize the stability of all of the fixed points of the maps:

a) $f(x) = \sqrt{x}$

b) $f(x) = 3x - x^3$

2. [5] *Superstable fixed points:*

Definition Let the map $x \mapsto f(x)$, have a stable fixed point $\bar{x} = 0$. The fixed point $\bar{x} = 0$ is superstable if

$$\lim_{n \to \infty} \frac{x_n}{\lambda^n} = 0$$

for all $\lambda \in (0, 1)$ and all orbits $\gamma(x_0)$ which converge to $\bar{x}$. Any other stable fixed point $\bar{x} \neq 0$ is superstable if $\bar{y} = 0$ is a superstable fixed point of the map $y \to F(y)$ where $F(y) = f(y + \bar{x}) - \bar{x}$.

Find an explicit formula for x_n in

$$x_{n+1} = f(x_n) = x_n^3$$

and then use it to show $\bar{x} = 0$ is superstable $\forall x_0 \in (0, 1)$. It suffices to show $\log\left(\frac{x_n}{\lambda^n}\right) \to -\infty$

3. [10] *Notation, Definitions and Period 2 Orbits:*

Let $f(x) = rx^2 - x$ define a map $x \mapsto f(x)$ for $r > 0$.

a) Determine the stability of all hyperbolic fixed points of f for all $r > 0$.

b) Does f have any non-hyperbolic fixed points. If so, for what r?

c) For $r = 2$, write out the first three terms of the orbits $\gamma\left(\frac{1}{4}\right)$ and $\gamma\left(\frac{1}{2}\right)$ in fractional form

d) For $r = 2$, what is $f([0, 1])$?

e) Does f have a period 2 orbit?

4. [20] The *quadratic map* $x \longmapsto f(x) = x^2 + c$ is another commonly studied (simple) map. Like the logistic map, the quadratic map has a period doubling bifurcation.

a) Draw qualitatively accurate cobweb diagrams for the cases $c = 0, \frac{1}{4}, 1$ and $x_0 = \frac{1}{4}$.

b) Determine explicit formulae for all fixed points $\bar{x}$ of f as a function of c. For what c do such fixed points exists and for which c are each linearly stable and unstable.

c) What type of bifurcation of fixed points occurs at $c = \frac{1}{4}$. What happens to orbits $\gamma(x_0)$ if $c > \frac{1}{4}$?

d) Letting $Q(x) = f(x) - x$, determine the quadratic $P(x)$ so that

$$f^2(x) - x = P(x)Q(x)$$

Using this factorization, determine an explicit formula for p_1, p_2 in the period two orbit

$$\gamma(p_1) = \{p_1, p_2, p_1, p_2, \ldots\}$$

Further, determine the range of c for which this periodic orbit exists and is linearly stable and unstable.

e) Use the results in a)-d) to sketch a (partial) bifurcation diagram for the map.