Homework 4 (Math 454) - 60max

Due: Friday, November 9, 2018.

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1. [5] Consider the initial value problem:

$$\frac{d\vec{x}}{dt} = A\vec{x} \qquad , \qquad \vec{x}(0) = \vec{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

One can find "a" fundamental matrix $\Psi(t)$ for the problem by putting any two independent solutions of the system into the columns of $\Psi(t)$. Then the general solution is $\vec{x}(t) = \Psi(t)\vec{c}$ where $\vec{c}$ is a constant vector. However, as show in my posted notes, there is a very special unique fundamental matrix $\Phi(t)$ given by

$$\Phi(t) \equiv \Psi(t)\Psi(0)^{-1}$$

such that the solution of the initial value problem is (simply)

$$\vec{x}(t) = \Phi(t)\vec{x}_0$$

Find $\Phi(t)$ for:

$$A = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix}$$

2. [35] Below are four different matrices A defining a linear system $\dot{x} = Ax$.

a) $A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$ **b)** $A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$

c) $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ **d)** $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

e) $A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$

For each of the systems

i) Classify the stability of the origin indicating if it is (Liapunov) stable, attracting, asymptotically stable, neutrally stable and/or unstable.

ii) Classify them as stable or unstable nodes/spirals, centers or saddles (when appropriate).

iii) Draw a qualitatively accurate phase portrait indicating eigenspaces E_{λ_k} for all real eigenvalues λ_k. Show the correct direction of the trajectories, and axes for any (elliptical) centers.

iv) Carefully define the stable, center, unstable manifolds $E^s(0)$ $E^c(0)$, and $E^u(0)$, respectively, of the origin. Some may be the empty set $\{\ \}$ and some may be the entire plane $\mathbb{R}^2$. Others may be $E^x(0) = span\{\zeta_k\}$ for some ζ_k.

3. [20] For the following systems,

i) Sketch the nullclines and indicate the direction of flow across them.

ii) Find all fixed points and classify them using linear stability analysis (saddle, stable node, center, etc.).

iii) For saddles, make sure your trajectories are tangent to the associated eigenvectors defining E^s and E^u.

iv) Lastly, draw a "plausible" phase portrait, i.e, consistent with the information previously obtained.

a)

$$\dot{x} = y \tag{1}$$
$$\dot{y} = x(x^2 - 1) \tag{2}$$

b)

$$\dot{x} = x^2 - y \tag{3}$$
$$\dot{y} = x - y \tag{4}$$