Homework 7 (Math 455) **Due: February 15-16, 2019.**

---

1. [10] Use the Poincare-Bendixson Theorem to prove that the planar system defined by

$$\begin{aligned}\dot{x} &= f_1(x,y) = x - y - x^3 \\ \dot{y} &= f_2(x,y) = x + y - y^3\end{aligned}$$

has at least one periodic orbit. Choose an annular trapping region M. The outer circle is ∂M_o of radius r_o. The inner circle is ∂M_i with radius r_i. The sole fixed point is the origin.

a) Show the fixed point is an unstable spiral so that flow is into M if r_i sufficiently small.

b) On ∂M_o an inward normal vector is $\vec{N} = (-x, -y)$. Show that on ∂M_o

$$\vec{N} \cdot (f_1, f_2) = -r^2 + r^4 F(\theta)$$

for some function $F(\theta)$ in polar coordinates.

c) Show $\frac{1}{2} \leq F(\theta) \leq 1$ and use this to find an r_o value such than $r > r_o \Rightarrow \vec{N} \cdot f > 0$.

2. [10] Use the Poincare-Bendixson Theorem to prove that the <u>perturbed</u> planar system defined by

$$\begin{aligned}\dot{x} &= f_1(x,y) = x - y - x(x^2 + y^2) + \epsilon x \\ \dot{y} &= f_2(x,y) = x + y - y(x^2 + y^2)\end{aligned}$$

has at least one periodic orbit for sufficiently small ϵ. Choose an annular trapping region M. The outer circle is ∂M_o of radius r_o. The inner circle is ∂M_i with radius r_i. The sole fixed point is the origin.

a) Show the origin is unstable (non-saddle) for all $\epsilon > 0$ so that (certainly) for sufficiently small ϵ flow across ∂M_i is into M.

b) Convert the system into polar coordinates (posted Notes17) and show that when $\epsilon = 0$ the system is

$$\dot{r} = r(1 - r^2) \qquad \dot{\theta} = 1$$

when $\epsilon \neq 0$ you should get

$$\dot{r} = r\left(\epsilon \cos^2\theta + 1 - r^2\right) \qquad \dot{\theta} = 1 - \frac{1}{2}\epsilon \sin(2\theta)$$

This implies the unperturbed ($\epsilon = 0$) system has a stable limit cycle $r = 1$.

c) Using your results in a)-b), give a carefully stated argument why M is a trapping region for $\epsilon > 0$ small enough.

3. [10] For each of the following systems draw phase portraits for the cases $\mu < 0, \mu = 0, \mu > 0$ and then draw a bifurcation diagram of the x coordinate of the fixed point branches versus μ.

$$\begin{aligned} \dot{x} &= \mu x - x^2 \quad \dot{y} = -y \\ \dot{x} &= \mu x + x^3 \quad \dot{y} = -y \end{aligned}$$

What types of bifurcations are these?

4. [10] (Bistability) Consider the following planar system where μ is a parameter.

$$\begin{aligned} \dot{x} &= f_1(x, y) = \phi(x) - \mu \\ \dot{y} &= f_2(x, y) = -y \end{aligned}$$

where $\phi(x)$ is a smooth function of x.

a) Fixed points are

$$X(\mu) = \begin{pmatrix} \bar{x}(\mu) \\ 0 \end{pmatrix}$$

where $\bar{x}$ are roots of $\phi(x) - \mu = 0$. Use linear stability analysis to show the stability of such fixed points depends only on the sign of the derivative $\phi'(x)$

b) Let $\phi(x) = x - x^3$ and plot μ as a function of x This is the locus of all fixed points. Rotate the figure and use the results in a) to draw a bifurcation diagram of x versus μ labeling branch stability.

c) For what μ is the system "bistable" - has two stable fixed points. Here I want an interval, i.e., $\mu \in (\mu_-, \mu_+)$.