Math 454 (2016) Assignment 1
(Due: Friday September 16, 2016 in class)

1. (5) Rewrite the following scalar differential equation as systems of autonomous differential equations and then state whether the resulting systems are linear or nonlinear.

\[ \frac{d^2x}{dt^2} - \frac{dx}{dt} + x = 1 \]

2. (5) Find two solutions of the initial value problem

\[ \dot{x} = \sqrt{1 - x^2}, \quad x(0) = 1 \]  

For each of your solutions state the maximum range of \( t \) for which they exist.

3. (15) For the following differential equations use phase portraits to classify all fixed points (as stable or unstable). If the equilibria is globally stable or globally unstable state so.
   a) \( \dot{x} = (1 - x)(2 - x) \)
   b) \( \dot{x} = x^2(6 - x) \)
   c) \( \dot{x} = e^{-x^2} - x \) (Use calculus to prove your result)

4. (5) Using Picard iteration find a (infinite) series representation of the solution to

\[ \dot{x} = x, \quad x(0) = 1. \]

Does the series converge to the unique solution \( x(t) = e^t \)?

5. (5) Let \( x(t) \) be the solution of

\[ \dot{x} = x^2 - 1, \quad x(0) = 1. \]

Compute the first four terms of the Taylor series of \( x(t) \) about \( t = 0 \), i.e.,

\[ x(t) = x(0) + x'(0)t + \frac{1}{2}x''(0)t^2 + \frac{1}{3}x'''(0)t^3 + \cdots \]

6. (5) Compute the potential function \( V(x) \) for the system

\[ \dot{x} = r + x - \frac{1}{3}x^3 \]

where \( r \in \mathbb{R} \) is a parameter. Sketch \( V(x) \) if \( r \in [0, 2/3] \) labelling the location of fixed points and categorizing them as stable or unstable.