Math 454 (2021) Assignment 1

(Due: Thursday, September 16, 2021 in class)

1. (5) The following initial value problem has the sole fixed point $\bar{x} = -1$. How long does the solution take to reach $\bar{x}$, i.e., infinite time, finite time (in which case exactly how much time):

$$\frac{dx}{dt} = -\sqrt{x+1} \quad , \quad x(0) = 3$$

2. (5) Find two solutions of the initial value problem

$$\dot{x} = \sqrt{1-x^2} \quad , \quad x(0) = 1 \tag{1}$$
$$\tag{2}$$

For each of your solutions state the maximum range of t for which they exist.

3. (15) For the following differential equations use phase portraits to classify all fixed points (as stable or unstable). Sketch a qualitatively accurate phase portrait labelling all fixed points.

a) $\dot{x} = (1-x)(2-x)$

b) $\dot{x} = x^2(6-x)$

c) $\dot{x} = e^{-x^2} - x$ (Use the Intermediate Value Theorem to prove existence)

4. (5) Using Picard iteration find a (infinite) series representation of the solution to

$$\dot{x} = x \quad , \quad x(0) = 1.$$

Does the series converge to the unique solution $x(t) = e^t$?

5. (5) Let $x(t)$ be the solution of

$$\dot{x} = x^2 - 1 \quad , \quad x(0) = 1.$$

Compute the first four terms of the Taylor series of $x(t)$ about $t = 0$, i.e.,

$$x(t) = x(0) + x'(0)t + \frac{1}{2}x''(0)t^2 + \frac{1}{3}x'''(0)t^3 + \cdots$$

6. (5) Compute the potential function $V(x)$ for the system

$$\dot{x} = r + x - \frac{1}{3}x^3$$

where $r \in \mathbb{R}$ is a parameter. Sketch $V(x)$ if $r \in [0, 2/3)$ labelling the location of fixed points and categorizing them as stable or unstable.