Homework 4 (Math 454) - 40max

Due: Thursday, November 4, 2021.

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1. [5] Consider the initial value problem:

$$\frac{d\vec{x}}{dt} = A\vec{x} \qquad , \qquad \vec{x}(0) = \vec{x}_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

One can find "a" fundamental matrix $\Psi(t)$ for the problem by putting any two independent solutions of the system into the columns of $\Psi(t)$. Then the general solution is $\vec{x}(t) = \Psi(t)\vec{c}$ where $\vec{c}$ is a constant vector. However, as show in my posted notes, there is a very special unique fundamental matrix $\Phi(t)$ given by

$$\Phi(t) \equiv \Psi(t)\Psi(0)^{-1}$$

such that the solution of the initial value problem is (simply)

$$\vec{x}(t) = \Phi(t)\vec{x}_0$$

Find $\Phi(t)$ for:

$$A = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix}$$

2. [35] Below are four different matrices A defining a linear system $\dot{x} = Ax$.

$$\textbf{a)}\ A = \begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix} \qquad \textbf{b)}\ A = \begin{bmatrix} -3 & 2 \\ -1 & -1 \end{bmatrix}$$

$$\textbf{c)}\ A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \qquad \textbf{d)}\ A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$$\textbf{e)}\ A = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix}$$

For each of the systems

i) Classify the stability of the origin indicating if it is (Liapunov) stable, attracting, asymptotically stable, neutrally stable and/or unstable.

ii) Classify them as stable or unstable nodes/spirals, centers or saddles (when appropriate).

iii) Draw a qualitatively accurate phase portrait indicating eigenspaces E_{λ_k} for all real eigenvalues λ_k. Show the correct direction of the trajectories, and axes for any (elliptical) centers.

iv) Carefully define the stable, center, unstable manifolds $E^s(0)$ $E^c(0)$, and $E^u(0)$, respectively, of the origin. Some may be the empty set $\{\ \}$ and some may be the entire plane $\mathbb{R}^2$. Others may be $E^x(0) = span\{\zeta_k\}$ for some ζ_k.