Homework 6 (Math 455) Due: Friday, January 25, 2019.

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1 [4] The following system $\dot{\mathbf{x}} = f(\mathbf{x})$ has a nonhyperbolic fixed point at the origin. By drawing the vector field on the unit circle deduce the index of this fixed point.

$$\dot{x} = x^2 - y^2 \quad , \quad \dot{y} = xy$$

2. [5] Recall that for a system

$$\begin{aligned} \dot{x} &= f_1(x, y) \\ \dot{y} &= f_2(x, y) \end{aligned}$$

the index $I(\Gamma)$ for some simple closed curve Γ can be computed by the integral formulation

$$I(\Gamma) = \frac{1}{2\pi} \int_\Gamma \frac{XdY - YdX}{X^2 + Y^2}$$

where

$$X(\theta) = f_1(x(\theta), y(\theta)) \quad , Y(\theta) = f_2(x(\theta), y(\theta))$$

for some parameterization $(x(\theta), y(\theta))$ of the curve Γ. Use the integral formulation of the index above to compute the index of the fixed point of the system

$$\begin{aligned} \dot{x} &= x \cos\alpha - y \sin\alpha \\ \dot{y} &= x \sin\alpha + y \cos\alpha \end{aligned}$$

where $\alpha \in [0, \pi]$. Choose Γ to be the unit circle.

3. [6] Use index theory to show that the following system has no closed orbits.

$$\begin{aligned} \dot{x} &= x(4 - y - x^2) \\ \dot{y} &= y(1 - x) \end{aligned}$$

Hints: Aside from finding the indices of the (four) fixed points you need to note the x and y axes are trajectories - hence periodic orbits cannot cross them. Draw a partial phase portrait containing the nullclines, fixed points (and their index) and write a clear argument why the system has no periodic orbits.

4. [5] Show that the planar system defined by

$$\begin{aligned} \dot{x} &= f_1(x, y) = -y^2 + 3x^2 \\ \dot{y} &= f_2(x, y) = -2xy - 2y \end{aligned}$$

can have no periodic orbits by showing it is a gradient system. Specifically, find a potential function V such that $\mathbf{f} = (f_1, f_2) = -\nabla V$.

5. [5] Show that the planar system defined by

$$\begin{aligned} \dot{x} &= f_1(x, y) = -x + 2y^3 - 2y^4 \\ \dot{y} &= f_2(x, y) = -x - y + xy \end{aligned}$$

can have no periodic orbits by finding a Liapunov function of the form $V = x^m + ay^n$ where the constants a, m, n are to be determined.