1. [10] For each of the following systems draw phase portraits for the cases $\mu < 0$, $\mu = 0$, $\mu > 0$ and then draw a bifurcation diagram of the $x$ coordinate of the fixed point branches versus $\mu$.

\[
\begin{align*}
\dot{x} &= \mu x - x^2 \quad \dot{y} = -y \\
\dot{x} &= \mu x + x^3 \quad \dot{y} = -y
\end{align*}
\]

What types of bifurcations are these, i.e., saddle node, pitchfork or transcritical?

2. [5] Consider the following planar system (Gray-Scott model of a certain chemical system)

\[
\begin{align*}
\dot{u} &= f(u,v) = a(1-u) - uv^2 \\
\dot{v} &= g(u,v) = uv^2 - (a+k)v
\end{align*}
\]

a) By adding the equations show that if $(\bar{u}, \bar{v})$ is a fixed point, $\bar{u}$ must be a root of the cubic:

\[P(u) = (1-u)(\beta^2 u^2 - \beta^2 u + a)\]

where $\beta = \frac{a}{a+k}$. Show that for some certain $(a,k)$ values there are three real fixed points $\bar{u}_0, \bar{u}_\pm$. State conditions on $(a,k)$ such that all three exist.

b) Show that two of the fixed points coalesce (in a saddle node bifurcation) at a specific (a-dependent) value $k^*$ of $k$.

3. [5] Consider the system

\[
\begin{align*}
\dot{x} &= -y + f(x,y,\mu) \\
\dot{y} &= x + g(x,y,\mu)
\end{align*}
\]

where

\[
\begin{align*}
f(x,y,\mu) &= \mu x + xy^2 \\
g(x,y,\mu) &= \mu y - x^2
\end{align*}
\]

Show that a Hopf bifurcation occurs at $\mu = 0$.

4. [5] In the following system $\dot{x} = f(x)$ the parameters $a, b > 0$ and the sole fixed point is the origin

\[
\begin{align*}
\dot{x} &= ax + xy - y \\
\dot{y} &= x - by + y(x^2 + y^2)
\end{align*}
\]

a) Shade in the region in the (positive) $(a,b)$-plane for which $det Df(0) > 0$

b) Shade in the region in the (positive) $(a,b)$-plane for which $Tr Df(0) > 0$

c) Draw the line in the $(a,b)$-plane across which the system has a Hopf bifurcation
5. [15] For each of the following polar planar systems draw a bifurcation diagram in $\mu$ labelling the stability of fixed points and periodic orbits. Separately, compute the period of all periodic orbits when they exist.

a) \[ \dot{r} = r^2(\mu - r)(2\mu - r) \quad , \quad \dot{\theta} = 1 \]

b) \[ \dot{r} = r \left( (\mu - 2)^2 + (r - 1)^2 - 1 \right) \quad , \quad \dot{\theta} = \frac{1}{\sin^2(\theta) + 2} \]

c) \[ \dot{r} = r(1 - r) \quad , \quad \dot{\theta} = \mu + \cos(2\theta) \]