Homework 9 (Math 455) **Due: Friday, February 24, 2017.**

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1. [10] For each of the following systems draw phase portraits for the cases $\mu < 0, \mu = 0, \mu > 0$ and then draw a bifurcation diagram of the x coordinate of the fixed point branches versus μ.

$$\begin{aligned} \dot{x} &= \mu x - x^2 \quad \dot{y} = -y \\ \dot{x} &= \mu x + x^3 \quad \dot{y} = -y \end{aligned}$$

What types of bifurcations are these, i.e., saddle node, pitchfork or transcritical?

2. [5] Consider the following planar system (Gray-Scott model of a certain chemical system)

$$\begin{aligned} \dot{u} &= f(u,v) = a(1-u) - uv^2 \\ \dot{v} &= g(u,v) = uv^2 - (a+k)v \end{aligned}$$

a) By adding the equations show that if $(\bar{u}, \bar{v})$ is a fixed point, $\bar{u}$ must be a root of the cubic:

$$P(u) = (1-u)(\beta^2 u^2 - \beta^2 u + a)$$

where $\beta = \frac{a}{a+k}$. Show that for some certain (a,k) values there are three real fixed points $\bar{u}_0, \bar{u}_\pm$. State conditions on (a,k) such that all three exist.

b) Show that two of the fixed points coalesce (in a saddle node bifurcation) at a specific (a-dependent) value k^* of k.

3. [5] Consider the system

$$\begin{aligned} \dot{x} &= -y + f(x,y,\mu) \ , \\ \dot{y} &= x + g(x,y,\mu) \ , \end{aligned}$$

where

$$\begin{aligned} f(x,y,\mu) &= \mu x + xy^2 \ , \\ g(x,y,\mu) &= \mu y - x^2 \ . \end{aligned}$$

Show that a Hopf bifurcation occurs at $\mu = 0$.

4. [5] In the following system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ the parameters $a, b > 0$ and the sole fixed point is the origin

$$\begin{aligned} \dot{x} &= ax + xy - y \ , \\ \dot{y} &= x - by + y(x^2 + y^2) \ , \end{aligned}$$

a) Shade in the region in the (positive) (a,b)-plane for which $detD\mathbf{f}(0) > 0$

b) Shade in the region in the (positive) (a,b)-plane for which $TrD\mathbf{f}(0) > 0$

c) Draw the line in the (a,b)-plane across which the system has a Hopf bifurcation

5. [15] For each of the following polar planar systems draw a bifurcation diagram in μ labelling the stability of fixed points and periodic orbits. Separately, compute the period of all periodic orbits when they exist.

a)

$$\dot{r} = r^2(\mu - r)(2\mu - r) \quad , \quad \dot{\theta} = 1 \quad .$$

b)

$$\dot{r} = r\left((\mu - 2)^2 + (r - 1)^2 - 1\right) \quad , \quad \dot{\theta} = \frac{1}{sin^2(\theta) + 2} \quad .$$

c)

$$\dot{r} = r(1 - r) \quad , \quad \dot{\theta} = \mu + cos(2\theta) \quad .$$