Math 454-455 Dynamical Systems xx
Supplementary Lecture Notes

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1 Introduction to Autonomous Systems

In this section we will introduce some notations applicable for later material involving differential equations and difference equations (and maps). Let $x = (x_1, x_2, \ldots x_n) \in \mathbb{R}^n$ and $t \in \mathbb{R}$ be time. Further we shall use the notation

$$\dot{x} = \frac{dx}{dt} = \left(\frac{dx_1}{dt}, \frac{dx_2}{dt}, \ldots \frac{dx_n}{dt}\right)$$

to denote derivatives. Thus, $\ddot{x} = \frac{d^2x}{dt^2}$. The *order* of a differential equation is determined by the order of the highest derivative. Thus, if $x(t) \in \mathbb{R}$,

$$\frac{d^3x}{dt^3} + x^3 = 0$$

is a third order (scalar) differential equation. To determine the order of a system of coupled differential equations you sum the maximum orders of all the derivatives of all the dependent variables. For example, consider the system

$$\frac{d^2x}{dt^2} - x\frac{d^2y}{dt^2} + y = 0 \qquad (1)$$

$$\frac{d^3x}{dt^2} - x\frac{dy}{dt} + x^2 = 0. \qquad (2)$$

In the system, there are at most third derivatives of $x(t)(n = 3)$ and second derivatives of $y(t)(m = 2)$ so the order of the *system* is $n + m = 5$.

We note that many scalar differential equations can be written as a system first order differential equations. This is done by introducing new dependent variables.

Example: If $y(t) \in \mathbb{R}$ is a solution of

$$\ddot{y} = g(y(t), \dot{y}(t))$$

then by defining $x_1 = y, x_2 = \dot{y}$ this second order equation can be written as the system

$$\dot{x} = f(x) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_2 \\ g(x_1, x_2) \end{pmatrix}.$$

If the independent variable t occurs explicitly the equation is said to be *nonautonomous*. If t does not occur explicity, the equation is *autonomous*.

Example: The equation

$$\dot{y}^2 + y^2 - \ddot{y} = 0$$

is autonomous whereas

$$\dot{y}^2 + t^2 y^2 - \ddot{y} = 0$$

and

$$\dot{y} = y^2 - t$$

are nonautonomous.

A system of ODEs can always be made autonomous by introducing an extra (trivial) dependent variable $y(t) = t$. By doing this one increases the order of the system:

Example: To convert the equation

$$\ddot{y} + e^t y^2 - \dot{y} = 0$$

into an autonomous system, introduce the dependent variables

$$x_1 = y \quad , \quad x_2 = \dot{y} \quad , \quad x_3 = t.$$

Then

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= x_2 - x_1^2 e^{x_3} \\ \dot{x_3} &= 1. \end{aligned}$$

Note that the system above can be written $\dot{x} = f(x)$, $x = (x_1, x_2, x_3)$ *if we define the vector-valued function* $f : \mathbb{R}^3 \to \mathbb{R}^3$ *by:*

$$f(x) = \begin{pmatrix} f_1(x_1, x_2, x_3) \\ f_2(x_1, x_2, x_3) \\ f_3(x_1, x_2, x_3) \end{pmatrix} = \begin{pmatrix} x_2 \\ x_2 - x_1^2 e^{x_3} \\ 1 \end{pmatrix}.$$

Note that a system

$$\dot{x} = f(x, t) \quad , \quad x \in \mathbb{R}^n,$$

is autonomous if $\frac{\partial f}{\partial t} = 0$, i.e., $\frac{\partial f_k}{\partial t} = 0, k = 1, 2, \dots n$ for each of the components of f [1]

Example: Let $y(t)$ and $z(t)$ be solutions of the system

$$\begin{aligned} \frac{d^3y}{dt^3} + t^2 y - \frac{dz}{dt} &= 0 \\ \frac{d^2z}{dt^2} + y - \frac{dy}{dt} &= 0. \end{aligned}$$

To re-write the system above as an autonomous system of first order equations, let

$$x_1 = y, \quad x_2 = \dot{y}, \quad x_3 = \ddot{y}, \quad x_4 = z, \quad x_5 = \dot{z}, \quad x_6 = t.$$

Then,

$$\frac{dx}{dt} = f(x) = \begin{pmatrix} x_2 \\ x_3 \\ x_5 - x_1 x_6^2 \\ x_5 \\ x_2 - x_1 \\ 1 \end{pmatrix}$$

is a 6^{th} order autonomous system.

Some differential equations cannot be written in the form $\dot{x} = f(x)$ for a real valued function f. A trivial example is:

$$\dot{x}^2 - x^2 = 0 \quad x \in \mathbb{R},$$

since solving for $\dot{x}$ yields $f(x) = \pm\sqrt{x}$, i.e. *two* functions for $x > 0$ (not to mention the complication that negative $x(t)$ can also be solutions).

For the remainder of this course we shall only examine autonomous systems of the form

$$\dot{x} = f(x) \quad x \in \mathbb{R}^n \quad n \leq 3$$

where f is continuously differentiable in the components x_i of x.

Lastly, we state a definition for linearity of systems of first order differential equations.

Definition 1 *If $\mathbf{x} = (x_1, x_2, \dots x_n)$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ the first order system of differential equations $\dot{\mathbf{x}} = f(\mathbf{x})$ is said to be <u>linear</u> if for some*

[1] which presumes a certain smoothness of the function f

matrix $A(t) \in \mathbb{R}^{n\times n}$ *and vector* $b(t) \in \mathbb{R}^n$*, the vector valued function* f *can be written*

$$f(\mathbf{x}) = A(t)\mathbf{x} + b(t) \ .$$

If neither A *nor* b *depends explicitly on* t *the system is said to be autonomous. If* $b = 0$ *the system is said to be homogeneous.*

Thus, the system

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & t \\ e^t & t \end{bmatrix} \mathbf{x}$$

is a second order linear homogeneous nonautonomous system whereas

$$\dot{\mathbf{x}} = \begin{bmatrix} 1 & 3 & 0 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{bmatrix} \mathbf{x}$$

is a third order linear homogeneous autonomous system.

2 First Order Equations

First order autonomous initial value problems can (in principle) be solved using separation of variables. For instance, the solution $x(t)$ of

$$\dot{x} = f(x) \quad , \quad x(0) = x_0$$

can be solved *implicitly* noting that

$$\int_{x_0}^{x(t)} \frac{ds}{f(s)} = t.$$

It is not always possible to invert this expression to find an *explicit* formula for $x(t)$. For example, applying this procedure to

$$\dot{x} = (x-1)(x+1)(x-2) \quad , \quad x(0) = x_0$$

yields the expression

$$ln\left(\frac{(x+1)^{1/6}(x-2)^{1/3}}{(x-1)^{1/2}}\right) \mid_{x_0}^{x(t)} = t$$

which cannot be inverted explicitly. In the absence of a simple formula for the solution, one may want easier techniques to determine qualitative aspects of the solution such as if they are bounded, increase or approach certain values as $t \to \infty$, to name a few.

Definition 2 *A fixed point $\bar{x}$ of*

$$\dot{x} = f(x) \quad , \quad x \in \mathbb{R}$$

is any value for which $f(\bar{x}) = 0$.

Fixed points are also called equilibria, steady states and critical points. Note that if $\bar{x}$ is a fixed point then $x(t) = \bar{x}$ is a solution of the initial value problem

$$\dot{x} = f(x) \quad , \quad x(0) = \bar{x}$$

for all $t > 0$.

Example: $\dot{x} = x^2 - 4x + 3 = (x-1)(x-3) = f(x)$ has two fixed points $\bar{x} = 1, 3$.

Example: $\dot{x} = f(x) = e^x + x$ has a sole fixed point $\bar{x}$. To prove this fact, one would appeal to the intermediate value theorem. In particular, f is continuous on $\mathbb{R}$ and $f \to \pm\infty$ as $x \to \pm\infty$ so that f must have at least one root. But, since $f'(x) = e^x + 1 > 0$ for all x, the root must be unique. Numerically, $\bar{x} \simeq -0.567$.

Definition 3 *A fixed point $\bar{x}$ of*

$$\dot{x} = f(x) \quad , \quad x \in \mathbb{R}$$

is said to be <u>*stable*</u> *if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|x_0 - \bar{x}| < \delta$ implies the solution $x(t)$ of*

$$\dot{x} = f(x) \quad , \quad x(0) = x_0$$

satisfies $|x(t) - \bar{x}| < \epsilon$ for all $t \geq 0$. If $\bar{x}$ is not stable, it is said to be <u>*unstable*</u>*.*

In words this definition states that if the initial value x_0 is sufficiently close to the fixed point then the solution $x(t)$ will remain close for all time. That should be distinguished from the concept of asymptotic stability:

Definition 4 *A fixed point $\bar{x}$ of*

$$\dot{x} = f(x) \quad , \quad x \in \mathbb{R}$$

is said to be <u>*asymptotically stable*</u> *it is stable and if there exists a $\delta > 0$ such that for every x_0 with $|x_0 - \bar{x}| < \delta$ the solution $x(t)$ of*

$$\dot{x} = f(x) \quad , \quad x(0) = x_0$$

approaches $\bar{x}$, i.e., $x(t) \to \bar{x}$ as $t \to \infty$.

which implies the solution $x(t)$ gets closer to $\bar{x}$ in time. For one dimensional systems where f is continuous, these notions are most often identical. A trivial example illustrating the difference would be if $f(x) = 0$ for all $x \in \mathbb{R}$. Then, every point $\bar{x} \in \mathbb{R}$ is a stable fixed point but none are asymptotically stable. In higher dimensions, differences in these definitions can be more subtle.

Definition 5 *If every solution $x(t)$ of*

$$\dot{x} = f(x) \quad , \quad x(0) = x_0 \in \mathbb{R}$$

approaches $\bar{x}$ as $t \to \infty$ and $f(\bar{x}) = 0$, then $\bar{x}$ is said to be a <u>*globally stable*</u> *fixed point.*

In practice one decides the stability of fixed points (for one dimensional systems) by plotting the <u>phase portrait</u> for the system. In a phase portrait, one plots x versus $\dot{x}$. Since $\dot{x} = f(x)$, the sign of f determines regions where $x(t)$ is increasing or decreasing in t.

Examples: in class

To make such phase plane arguments rigorous one needs to appeal to the following Lemma (in [5]) [2].

Lemma 1 *$\bar{x}$ is an asymptotically stable fixed point of*

$$\dot{x} = f(x) \quad ,$$

if and only if there is a $\delta > 0$ such that

$$0 < |x - \bar{x}| < \delta \Rightarrow (x - \bar{x})f(x) < 0.$$

Note that the later statement can be expanded into the two statements

$$\begin{array}{ll} f(x) < 0 \quad if & x \in (\bar{x}, \bar{x} + \delta) \\ f(x) > 0 \quad if & x \in (\bar{x} - \delta, \bar{x}). \end{array}$$

In words, if f is positive to the left of $\bar{x}$ and negative to the right, then $\bar{x}$ is asymptotically stable

2.1 Uniqueness of solutions

Proving that the initial value problem

$$\dot{x} = f(x) \quad , \quad x(0) = x_0$$

has a solution (existence of solution) or that the solution is unique is not an easy matter. Even if $f(x)$ is continuous, solutions may not be unique. To illustrate this, consider the problem

$$\dot{x} = \sqrt{x} = f(x) \quad , \quad x(0) = 0.$$

Here $f(x)$ is continuous on the closed interval $[0, \infty)$ but it is easily verified that

$$x(t) \equiv 0$$

and

$$x(t) = \frac{1}{4}t^2$$

[2] A partial proof will be presented later

are both solutions for all $t \geq 0$. In fact, the situation is much worse. There is a one parameter family of solutions:

$$x_\alpha(t) \equiv \begin{cases} 0 & 0 \leq t \leq \alpha \\ \frac{1}{4}(t-\alpha)^2 & t > \alpha \end{cases}$$

For every $\alpha > 0$, $x_\alpha(t)$ satisfies the differential equation and initial condition. This is true even at $t = \alpha$ since $\dot{x}_\alpha(\alpha) = 0$ is defined (note how the left and right derivatives are equal at $t = \alpha$).

Theorems which guarantee the existence of unique solutions are difficult to prove. The following one can be found in [2].

Theorem 1 *Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f = f(x,t)$. Assume that f and $\frac{\partial f}{\partial x}$ are continuous on the (closed) rectangular region*

$$R = \{(x,t) : |x - x_0| \leq b, t \in [0,a]\}.$$

Defining

$$M = \max_R |f(x,t)| \quad , \quad \alpha = \min\left\{a, \frac{b}{M}\right\},$$

then the initial value problem

$$\dot{x} = f(x,t) \quad , \quad x(0) = x_0,$$

has a unique solution for $0 \leq t \leq \alpha$.

This theorem says three things. A solution exists, it is unique and it persists for at least a finite time. Persistence we will discuss later. The proof of the existence of solutions requires generating a solution candidate via something called Picard iteration. Here we briefly describe this. If $f(x)$ is continuous then integrating $\dot{x} = f(x), x(0) = x_0$ yields the identity

$$x(t) = x_0 + \int_0^t f(x(s))ds.$$

Next, we define a sequence of functions $x_n(t)$ by the recursion

$$x_{n+1}(t) = x_0 + \int_0^t f(x_n(s))ds \quad , \quad n = 0, 1, 2, \ldots$$

where x_0 is the initial condition. This formula defines <u>Picard iteration</u>.

For an arbitary f, Picard iteration would yield:

$$x_1(t) = x_0 + \int_0^t f(x_0)ds = x_0 + f(x_0)t.$$

Likewise,

$$x_2(t) = x_0 + \int_0^t f(x_1(t))ds = x_0 + \int_0^t f(x_0 + f(x_0)s)ds.$$

The object is to show that $x_n(t) \to X(t)$ and then verify that the function $X(t)$ satisfies the differential equation. Uniqueness is proven separately. For details, see [2].

Example: For

$$\dot{x} = x^2 + 1 = f(x) \quad , \quad x(0) = x_0 = 0$$

Picard iteration yields

$$x_1(t) = 0 + \int_0^t f(0)ds = t.$$

Then

$$x_2(t) = \int_0^t f(x_1(s))ds = \int_0^t (s^2 + 1)ds = t + \frac{t^3}{3}.$$

Continuing,

$$x_3(t) = \int_0^t f\left(s + \frac{s^3}{3}\right) ds = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{1}{63}t^7$$

The exact solution is $x(t) = tan(t)$ whose Taylor series about $t = 0$ matches the first three terms of $x_3(t)$. In this manner, Picard iteration can be used to compute Taylor series approximations of the solution. In this specific case, though, note that the fourth term of $x_3(t)$ does not equal the fourth term of the Taylor series:

$$x(t) = t + \frac{1}{3}t^3 + \frac{2}{15}t^5 + \frac{17}{315}t^7 + \cdots.$$

Suppose $f(x)$ and its derivative $f'(x)$ are continuous on $\mathbb{R}$. Further assume that $f(x)$ is uniformly bounded on $\mathbb{R}$, i.e., there is a constant $K > 0$ such that $|f(x)| < K$ for all $x \in \mathbb{R}$. Then, using the Theorem, it is not hard to show that solutions of the initial value problem

$$\dot{x} = f(x) \quad , \quad x(0) = x_0$$

exist and are unique for all $t \geq 0$ (can you prove this?). This fact can be used to rigorously prove many of the facts about phase portraits. For example,

Lemma 2 *Suppose that $\bar{x}$ is a fixed point of $\dot{x} = f(x)$ and that*

$$(x - \bar{x})f(x) < 0 \quad for\ all\ x \in \mathbb{R}, x \neq \bar{x},$$

i.e., $\bar{x}$ is stable. Further suppose that the initial value problem

$$\dot{x} = f(x) \quad , \quad x(0) = x_0 < \bar{x}$$

has a unique solution $x(t)$ for all $t \geq 0$. Then, $x(t) \to \bar{x}$ as $t \to \infty$ (monotonically in t).

Proof: First note that if $x(T) = \bar{x}$ at some finite time $T < \infty$ then $x(t) = \bar{x}$ for all $t \geq T$. This is due to uniqueness of the solution. In particular, the sole solution of

$$\dot{x} = f(x) \quad , \quad x(T) = \bar{x}$$

is $x(t) = \bar{x}$. [3] *If this is the case then it is trivially true that $x(t) \to \bar{x}$ as $t \to \infty$. Regardless, we now know that $x(t)$ is bounded above by $\bar{x}$. Since $\dot{x} = f(x) > 0$ for all $x \in [x_0, \bar{x})$, $x(t)$ is a monotonically increasing function which is bounded above. Thus* [4]*, $x(t)$ must approach a limit $X \leq \bar{x}$ as $t \to \infty$. We claim that $X = \bar{x}$. Suppose not that $X < \bar{x}$ strictly. Then $\dot{x} \geq f(X) > \delta > 0$ for some δ and all $t \geq 0$. This implies $x(t) \geq \delta t + x_0$ and that $x(t)$ reaches $\bar{x}$ in finite time which is a contradiction.*

With a minor modification this proof can be adapted to show that $x(t) \to \bar{x}$ for $x_0 > \bar{x}$. Thus, we have shown that if the solution is *unique* and persists for all $t > 0$ that asymptotic stability and stability are the same. In essence, we have proven Lemma 1.

2.2 Linear Stability

If $f(x)$ is a continuously differentiable function and $f'(\bar{x}) < 0$ at a fixed point then we know that $f(x) > 0$ for x slightly smaller than $\bar{x}$ and $f(x) < 0$ for x slightly larger than $\bar{x}$. In other words, if $f'(\bar{x}) < 0$, $\bar{x}$ is stable.

Definition 6 *Let $\bar{x}$ be a fixed point of $\dot{x} = f(x)$. We say $\bar{x}$ is linearly stable if $f'(x)$ is continuous near $\bar{x}$ and $f'(\bar{x}) < 0$.*

[3] Note how this part of the proof indicates why you can't "move" through a place where $\dot{x} = 0$.

[4] using a result from real analysis

This definition must be used carefully. If $\bar{x}$ is not linearly stable it might be stable or possibly unstable. For example, if $f(x) = x^3$, $\bar{x} = 0$ is not linearly stable since $f'(0) = 0$ and the phase portrait implies it is unstable. In contrast, if $f(x) = -x^3$, $\bar{x} = 0$ is not linearly stable but is stable. When $f'(\bar{x}) = 0$, "linear analysis" fails and other means must be used to determine stability.

Note, however, if $f'(\bar{x}) < 0$ strictly then phase portraits would indicate $\bar{x}$ is unstable, i.e.,

Lemma 3 *If $\dot{x} = f(x), x \in \mathbb{R}$, $f(\bar{x}) = 0$ and $f'(x)$ is continuous near $\bar{x}$ then*

$$f'(\bar{x}) < 0 \Rightarrow \bar{x} \; is \; unstable$$

So, why is it called linear stability? Suppose that $\bar{x}$ is a fixed point of

$$\dot{x} = f(x) \quad , \quad x(0) = x_0 \tag{3}$$

Using Taylor's Theorem (under appropriate smoothness assumptions about f) we may write

$$\dot{x} = f(\bar{x}) + f'(\bar{x})(x - \bar{x}) + \frac{1}{2} f''(\bar{x})(x - \bar{x})^2 + \cdots$$

If we define $y(t) = x(t) - \bar{x}$ (noting $f(\bar{x}) = 0$ because $\bar{x}$ is a fixed point) then

$$\dot{y} = f'(\bar{x})y + \frac{1}{2} f''(\bar{x})y^2 + \cdots .$$

If we only keep the linear part of the right side we find

$$\dot{y} \simeq f'(\bar{x})y$$

How "equal" the two sides are will depend on how close the initial condition x_0 is to $\bar{x}$. It is also possible that the two sides are nearly equal for a small time interval but that after some time they are very different.

The equation

$$\dot{y} = f'(\bar{x})y \tag{4}$$

is called the linearization of (3) about $\bar{x}$. Some authors refer to (4) as the linear variational equation since y is the variation from $\bar{x}$. Notice that if $f'(\bar{x}) = 0$, the linearization would be $\dot{y} = 0$ or that $x(t)$ would not change in time. Clearly that would be a bad approximation. This is an instance when "linearization" fails and why one does not define linear stability for that case. The basic theorem involving linear stability analysis is (proven in [5]):

Theorem 2 *Suppose $\bar{x}$ is a fixed point of*

$$\dot{x} = f(x),$$

$f'(x)$ is defined and continuous near $x = \bar{x}$, and $f'(\bar{x}) < 0$ then $\bar{x}$ is asymptotically stable.

Proof: Here we give a proof for f that are twice continuously differentiable. First we set $y(t) = x(t) - \bar{x}$ so that

$$\dot{y} = f(y + \bar{x}).$$

It is then sufficient to show that $y(t) \to 0$ as $t \to \infty$ if $|y(0)|$ is sufficiently small. Taylor's theorem then implies that for every y there is a constant η_y which depends on (is a function of) y such that

$$f(y + \bar{x}) = f(\bar{x}) + f'(\bar{x})y + \frac{1}{2}f''(\eta_y)y^2.$$

Defining $g(y) = \frac{1}{2}f''(\eta_y)y^2$ and noting $f(\bar{x}) = 0$,

$$\dot{y} = F(y) \equiv f(y + \bar{x}) = f'(\bar{x})y + g(y). \tag{5}$$

From (5) it is evident that $g'(y)$ exists, is continuous and that $g(0) = g'(0) = 0$. Since $g'(y)$ is continuous, for every $\epsilon > 0$ there is a $\delta > 0$ such that $|y| < \delta \Rightarrow |g'(y)| < \epsilon$. Thus, from the identity

$$g(y) = \int_0^y g'(s)ds$$

we conclude that

$$|g(y)| \le \epsilon|y|$$

and that for $y \in (-\delta, +\delta)$

$$(f'(\bar{x}) - \epsilon)y \le F(y) \le (f'(\bar{x}) + \epsilon)y.$$

Therefore, for ϵ sufficiently small, the sign of $F(y)$ near the fixed point $y = 0$ is the same as $f'(\bar{x})$ making $y = 0$ a stable fixed point of (5). By Lemma 1, $y = 0$ is asymptotically stable proving the result.

2.3 Blowup

Some solutions exist but for only a finite time. One reason is that they *blowup*, i.e $|x(t)| \to \infty$ as $t \to T^-$ for some $T < \infty$. For instance, the solution of

$$\dot{x} = f(x) = x^2 + 1 \quad , \quad x(0) = 0$$

is $x(t) = tan(t)$ which blows up as $t \to \frac{\pi}{2}$. What is surprising about this example is that $f(x)$ is a perfectly simple and differentiable function.

Knowing for how long a solution exists is also not easy to prove apriori but Theorem 1 gives at least one way of estimating a lower bound α of the times for which a unique solution must exist. Ideally, one wants to find a and b for which α defined by

$$M = \max_R |f(x,t)| \quad , \quad \alpha = \min\left\{a, \frac{b}{M}\right\} \tag{6}$$

is as big as possible. It is easy to make α small by letting $a \to 0+$. So, without any loss of generality, we may as well choose

$$\alpha = \frac{b}{M} = \frac{b}{\max_{|x-x_0|\leq b} |f(x)|}$$

when the system is autonomous. This α is a function of b, i.e., $\alpha = \alpha(b)$. So long as a maximum M exists, then the biggest one could make this α is by choosing a b which maximizes it:

$$\alpha(b) \leq \alpha^* = \max_{b>0} \frac{b}{\max_{|x-x_0|\leq b} |f(x)|}.$$

Then, we are still assured that unique solutions exist for $t \in [0, \alpha^*]$. To illustrate an application of this we have the following example:

Example: Find an α^* for which Theorem 1 guarantees the existence of a unique solution of

$$\dot{x} = f(x) = x^2 + 1 \quad , \quad x(0) = x_0 = 0$$

for all $t \in [0, \alpha^*]$. First note that for any b, the maximum value of $|f(x)|$ on $[-b, b]$ is $M = b^2 + 1$. Thus,

$$\alpha^* = \max_{b>0} \frac{b}{b^2+1}.$$

Elementary calculus (or a plot) reveals that $G(b) = \frac{b}{b^2+1}$ has a maximum at $b = 1$ at which $G(1) = \frac{1}{2}$ so that a solution must exist for all $t \leq \alpha^* = \frac{1}{2}$. Note that the true solution $x(t) = tan(t)$ exists for t larger than this estimate, i.e., $t < \frac{\pi}{2}$. This emphasizes the fact that α^* is a lower bound only.

2.4 Comparsion Methods for Blowup

One way that you can show a solution blows up is to compare the solution with one that you know blows up. For example, suppose you know that the solution $x(t)$ of

$$\dot{x} = f(x) \quad , \quad x(0) = x_0$$

blows up at $t = T$. Then consider the separate problem

$$\dot{y} = g(y) \quad , \quad y(0) = x_0$$

noting it has the same initial condition. If $g(x) \geq f(x)$ for all x then $\dot{y} \geq \dot{x}$ for all t for which $x(t)$ and $y(t)$ exist. Since they have the same initial condition, $y(t) \geq x(t)$ which implies that y must blow up at some time $T^* \leq T$.

Stated more precisely we have the following:

Theorem 3 *Let $x(t)$ and $y(t)$ be solutions of the following initial value problems:*

$$\dot{x} = f(x) \quad , \quad x(0) = x_0$$
$$\dot{y} = g(y) \quad , \quad y(0) = x_0$$

and assume such solutions exist $\forall t \in [0, a]$. Then, if $g(x) \geq f(x)$ for all x,

$$y(t) \geq x(t) \quad , \quad \forall t \in [0, a] \, .$$

For a proof of this Theorem see Theorem 6.1 in [4]. Below we illustrate an application.

Example: Show that the solution of

$$\dot{y} = g(y) = y^4 + y^2 + 1 \quad , \quad y(0) = 1$$

blows up. First note that the solution of

$$\dot{x} = f(x) = x^2 + 1 \quad , \quad x(0) = 1$$

is $x(t) = tan(t + \frac{\pi}{4})$ which blows up at $T = \frac{\pi}{4}$. Since $g(x) \geq f(x)$ for all x, $y(t)$ must blow up at a time $T^* \leq \frac{\pi}{4}$.

We conclude this section by making some remarks about differential inequalities. In the aforementioned comparison arguments we appeal to the logic that if $\dot{x} \geq \dot{y}$ and $x(0) = y(0)$ then $x(t) \geq y(t)$ for all t for which solutions exist. A (generally) false converse type of argument is that $x(t) \geq y(t)$ implies $\dot{x} > \dot{y}$. One cannot differentiate both sides of an inequality and expect to retain the ordering. For example, let $x(t) = 2t - 2$ and $y(t) = t$. Then,

$$x(t) < y(t) \quad , \quad \forall t < 1$$

and,

$$\dot{x}(t) > \dot{y}(t) \quad , \quad \forall t < 1 \, .$$

2.5 Potential Functions

Another way of visualizing why some fixed points are stable while others are unstable is to use a potential function. If

$$\dot{x} = f(x)$$

then we define a potential function:

$$V(x) = -\int^x f(s) ds.$$

Then,

$$\dot{x} = -\frac{dV}{dx}.$$

For any solution $x(t)$ we have

$$\frac{d}{dt} V(x(t)) = \frac{dV}{dx}\frac{dx}{dt} = -f(x(t))^2 \leq 0$$

or that V decreases [5]. From its definition, fixed points occur where $V' = 0$, i.e., the slope is zero. Moreover, $V''(x) = -f'(x)$ so if V is (strictly) concave up at a fixed point, the fixed point is (linearly) stable. The local minima of V form "wells" in the potential where the solution $x(t)$ must then "fall" into.

Example: The potential function associated with $\dot{x} = x - x^3$ is

$$V(x) = \int^x (s - s^3) ds = \frac{1}{4} x^2 (1 - x^2).$$

A graph of $V(x)$ shows that V is concave up only at its sole local minima $x = 0$. This corresponds to the sole stable fixed point of $\dot{x} = x - x^3$.

[5] strictly speaking V is non-increasing since it could remain constant

3 Fundamental Bifurcations of $\dot{x} = f(x, \mu), x \in \mathbb{R}$

A question of interest is how the location and stability of fixed points of a system

$$\dot{x} = f(x, \mu), x \in \mathbb{R}$$

depends on a parameter (or parameters) μ. If such a system is a model of some physical, chemical or biological problem then it is particularly important to keep track of stable fixed points since those represent what the state variable would tend to as $t \to \infty$. In modelling situations, fixed point location is also important. For instance, most state variables (concentration, population, mass) are positive so only nonnegative fixed points are *physical*.

Roughly speaking, as one varies a parameter fixed points can coalesce with other fixed points, appear, disappear and change their stability. Values of (x, μ) at which such things occur are called *bifurcation points* [6]. There are of course many ways which these things can happen and they all depend on the definition of $f(x, \mu)$. But, generally speaking, there are three generic types of *bifurcations*: Saddle node, Transcritical and Pitchfork. Each of these types of bifurcations has a generic system which exemplifies their salient features. Respectively, these systems are

$$\begin{aligned} \dot{x} &= f(x, \mu) = x^2 + \mu && (7) \\ \dot{x} &= f(x, \mu) = \mu x - x^2 && (8) \\ \dot{x} &= f(x, \mu) = \mu x - x^3 && (9) \end{aligned}$$

In subsequent sections we shall discuss each of these bifurcations and theory related to identifying such bifurcations.

3.1 Saddle-Node Bifurcations

Consider the first order differential equation

$$\dot{x} = f(x, \mu) = \mu + x^2$$

where $\mu \in \mathbb{R}$ is a parameter. For $\mu > 0$ this equation has no fixed points. At $\mu = 0$ it has the sole (unstable) fixed point $\bar{x} = 0$. But for $\mu < 0$ it has two fixed points. Solving $f = 0$ for x one finds the two branches of fixed points:

$$\bar{x}_{\pm} = \pm\sqrt{-\mu} \quad , \quad \mu < 0.$$

[6] this is not a precise definition of a bifurcation point but will suffice for the moment

It is easily verified that the branch $\bar{x}_+$ are stable equilibria while the other branch $\bar{x}_-$ is unstable.

When the two branches are plotted as a function of μ the stable (node) and unstable (saddle) fixed points coalescese at a bifurcation point $(\mu^*, x^*) = (0, 0)$. The locus of fixed points in the (μ, x)-plane together with a labelling of the fixed points stability [7] constitute a bifurcation diagram. It must be noted that one need not solve $f = 0$ for x to be able to plot the locus of equilibria. Viewed another way $\mu = x^2$ gives the locus as a sole function of x, i.e., one need not compute "branches".

This particular bifurcation is called a saddle-node bifurcation. The name comes from the fact it is a one-dimensional analog of the bifurcation where a saddle and a stable node coalescece in a planar system. In some textbooks, authors refer to the bifurcation as one with a "quadratic tangency" because the locus when viewed as a function of x, $\mu = \mu(x)$ is locally quadratic near the bifurcation point (μ^*, x^*). By definition, at a saddle node bifurcation with quadratic tangency one must have

$$\frac{d\mu}{dx}(x^*) = 0, \tag{10}$$

$$\frac{d^2\mu}{dx^2}(x^*) \neq 0. \tag{11}$$

Example: Show that $\dot{x} = \mu - x + x^3$ has two saddle-node bifurcations. The locus of equilibria is given by $\mu = \mu(x) = x - x^3$. Solve $\mu'(x) = 1 - 3x^2 = 0$ yields $x^*_\pm = \pm\frac{1}{\sqrt{3}}$ Since $\mu''(x^*_\pm) = \mp 2\sqrt{3} \neq 0$, the condition (11) is satisfied and there are two saddle-node bifurcation points

$$(\mu^*_+, x^*_+) = \left(\frac{2}{9}\sqrt{3}, \frac{1}{\sqrt{3}}\right) \quad , \quad (\mu^*_-, x^*_-) = \left(-\frac{2}{9}\sqrt{3}, -\frac{1}{\sqrt{3}}\right).$$

3.2 Saddle Node Local Theory

Though the preceding introductory examples illustrate ways to detect saddle node bifurcations, alternate methods based on more general local theory are sometimes useful. At a practical level, it may not be possible to solve $f(x, \mu) = 0$ for x as a function of μ or μ as a function of x. In this section we introduce theory which leads to a theorem which can be used in such settings. To begin we introduce the following defintion:

[7] solid line for stable, dashed lines for unstable

Definition 7 *A fixed point $\bar{x}$ of $\dot{x} = f(x, \mu)$ is* *hyperbolic* *if* $\frac{\partial f}{\partial x}(\bar{x}, \mu) \neq 0$.

Hyperbolicity of fixed points is at the very heart of bifurcation theory. In latter sections we will show that all the basic bifurcation points (saddle node, transcritical, pitchfork) correspond to nonhyperbolic fixed points.

Next, we will adopt subscript notation for partial derivatives. For example $f_x = \frac{\partial f}{\partial x}, f_\mu = \frac{\partial f}{\partial \mu}, f_{x\mu} = \frac{\partial^2 f}{\partial x \partial \mu}$, etc. Thus, $\bar{x}$ is hyperbolic if $f_x(\bar{x}, \mu) \neq 0$.

In instances where it is not possible to find explicit expressions for the locus of equilibria one may still want to decide if a bifurcation point is a saddle-node bifurcation. Here we will develop a Taylor series decription of saddle-node bifurcations based on the idea of quadratic tangency.

Suppose that on the locus of equilibria $\mu = \mu(x)$, i.e. μ is a function of x in a neighbourhood of the suspect bifurcation point. If the bifurcation occurs at (μ^*, x^*) then by letting

$$X = x - x^* \quad , \quad \eta = \mu - \mu^*$$

the differential equation for X is

$$\dot{X} = F(X, \eta) = f(X + x^*, \eta + \mu^*)$$

which has a bifurcation at $(X, \eta) = (0, 0)$. Thus, without any loss of generality we will assume that the bifurcation occurs at $(\mu, x) = (0, 0)$.

In this setting $f(0, 0) = 0$, $\mu(0) = 0$ and

$$f(x, \mu(x)) = 0$$

for all x near the bifurcation point. Differentiating this expression in x yields

$$f_x(x, \mu(x)) + f_\mu(x, \mu(x))\frac{d\mu}{dx} = 0 \tag{12}$$

which when solved for $\mu'(x) = \frac{d\mu}{dx}$ gives

$$\mu'(x) = -\frac{f_x(x, \mu(x))}{f_\mu(x, \mu(x))}.$$

One condition which must be satisfied at a quadratic tangency is that $\mu'(0) = 0$. Given the formula above, this can only happen if

$$f_x(0, 0) = 0 \tag{13}$$

$$f_\mu(0, 0) \neq 0. \tag{14}$$

Together, the conditions (13)-(14) imply that $x = 0$ is a nonhyperbolic fixed point. For there to be a quadratic tangency, however, we also need to have $\mu''(0) \neq 0$. By differentiating (12) in x, evaluating the resulting expression at $x = 0$ and using (13) it can be shown that

$$f_{xx}(0,0) + f_\mu(0,0)\mu''(0) = 0.$$

Because $f_\mu(0,0) \neq 0$, $\mu''(0) \neq 0$ only if

$$f_{xx}(0,0) \neq 0. \tag{15}$$

In conclusion, a saddle-node bifurcation with quadratic tangency will exist if the three conditions (13)-(15) are satisfied. We can also deduce that the Taylor series expansion of f about such a bifurcation point will have the form

$$f(x,\mu) = a_0\mu + a_1x^2 + a_2x\mu + a_3\mu^2 + O(3)$$

for some constants $a_0 \neq 0$,$a_1 \neq 0$, a_2 and a_3. Here, $O(3)$ is notation to indicate higher order terms in the Taylor series, i.e., $x^3, x^2\mu, \ldots$. In the next section we will discuss how one can simplify this expression to create what is called the "normal form" for the bifurcation.

Theorem 4 *Let $\dot{x} = f(x,\mu)$ and assume that for all (μ, x) near some point (μ^*, x^*) f has continuous (mixed) derivatives up to and including third order, i.e., $f_x, f_\mu, f_{xx}, \ldots, f_{x\mu\mu}, f_{\mu\mu\mu}$. If*

$$\begin{aligned} f(x^*,\mu^*) &= 0 & (16)\\ f_x(x^*,\mu^*) &= 0 & (17)\\ f_\mu(x^*,\mu^*) &\neq 0 & (18)\\ f_{xx}(x^*,\mu^*) &\neq 0 & (19)\end{aligned}$$

then $\dot{x} = f(x,\mu)$ has a saddle-node bifurcation with quadratic tangency at (μ^*, x^*).

Example: Consider

$$\dot{x} = f(x,\mu) = \mu - x - e^{-x}.$$

A nonhyperbolic equilibria exists at any pair of (μ, x) such that

$$\begin{aligned} f = 0 \quad &\Leftrightarrow \quad \mu - x - e^{-x} = 0\\ f_x = 0 \quad &\Leftrightarrow \quad -1 + e^{-x} = 0\end{aligned}$$

Thus $(\mu^*, x^*) = (1, 0)$ *is a bifurcation point. Since* $f_{xx}(x, \mu) = -e^{-x}$*,* $f_{xx}(0, 1) = -1 \neq 0$*. Lastly,* $f_\mu = 1 \neq 0$ *so that a saddle-node bifurcation of quadratic tangency occurs at* $(\mu^*, x^*) = (1, 0)$*, i.e.,*

$$f = 0 \quad , \quad f_x = 0 \quad , \quad f_\mu \neq 0 \quad , \quad f_{xx} \neq 0.$$

The (2-variable) Taylor series of $f(x, \mu)$ *about* $(\mu^*, x^*) = (1, 0)$ *is*

$$f(x, \mu) = (\mu - 1) - \frac{1}{2}x^2 + \frac{1}{6}x^3 + O(x^4)$$

so that

$$\dot{x} = \eta - \frac{1}{2}x^2 + O(x^3)$$

where $\eta = \mu - 1$ *is a new parameter. For this example, the equation above is in "normal form".*

Note that some saddle-node bifurcations do not result from "quadratic" tangencies. For example,

$$\dot{x} = \mu + x^4$$

has a saddle-node bifurcation at $(\mu, x) = (0, 0)$ even though $f_{xx}(0, 0) = 0$. This is an example of a "quartic" tangency. Clearly other more complicated variants can occur as well. For example, just consider what happens at $(\mu, x) = (0, 0)$ if

$$\dot{x} = (x^2 - \mu)(x^2 - 4\mu).$$

Also, some bifurcations that are quadratic are not a result of a saddle-node bifurcation. For example, consider

$$\dot{x} = f(x, \mu) = \sqrt{\mu} - x \quad , \quad \mu \geq 0,$$

which has a sole branch of fixed points $x = \sqrt{\mu}$ and a quadratic tangency at $x = 0$. At $(\mu^*, x^*) = (0, 0)$ two branches of fixed points do not coalescece. This example does not violate Theorem 4 since $f_\mu(0, 0)$ is not defined, let alone continuous for all (μ, x) near $(\mu^*, x^*) = (0, 0)$.

Example: Consider a minor variant of the previous example:

$$\dot{x} = f(x, \mu) = \mu - x - e^{-\mu x}.$$

We will show that this problem has two bifurcations with quadratic tangencies. Thus, we want to show that there are two pairs (μ, x) *that* *satisfy:*

$$\begin{aligned} f(x,\mu) &= \mu - x - e^{-\mu x} = 0 && (20)\\ f_x(x,\mu) &= -1 + \mu e^{-\mu x} = 0 && (21)\\ f_\mu(x,\mu) &= 1 + x e^{-\mu x} \neq 0 && (22)\\ f_{xx}(x,\mu) &= -\mu^2 e^{-\mu x} \neq 0 && (23) \end{aligned}$$

Equations (20)-(21) are used to locate the bifurcation point (i.e., and that the fixed point be "nonhyperbolic") whereas the condition (22)-(23) are used to verify that there is a quadratic tangency in the bifurcation diagram.

Note that (21) implies

$$e^{-\mu x} = \frac{1}{\mu} \qquad (24)$$

so that if a solution (μ, x) exists it must have $\mu > 0$. Using (24) in (20) one finds

$$x = x^*(\mu) = \mu - \frac{1}{\mu} \quad , \quad \mu > 0.$$

So if $\mu = \mu^*$ at the bifurcation point the associated value of $x = \mu^* - 1/\mu^*$ [8]. Using this in (21) we conclude that μ^* must be a root of

$$g(\mu) = f_x(x^*(\mu), \mu) = \mu e^{1-\mu^2} - 1 \quad , \quad \mu > 0.$$

Next we will verify that $g(\mu)$ has two positive roots. First note that $g(0) = -1$ and $g(\mu) \to -1$ as $\mu \to \infty$. Thus, it suffices to determine the location of maxima and minima of g to determine the number of roots it has. From

$$g'(\mu) = e^{1-\mu^2}\left(1 - 2\mu^2\right),$$

it is evident that $g'(\mu) = 0$ at $\mu = \mu^\pm = \pm\frac{1}{\sqrt{2}}$. Of these values only $\mu^+ > 0$. Since $g(\mu^+) = \frac{1}{\sqrt{2}}e^{1/2} - 1 \simeq 0.17 > 0$, g has a sole positive maximum at $\mu = mu^+$. By the intermediate value theorem, g must have two roots μ_1^* and μ_2^* with $0 < \mu_1^* < \mu^+ < \mu_2^*$. By inspection $\mu_2^* = 1$. The other value is (numerically) $\mu_1^* \simeq 0.45$. In conclusion, there are only two nonhyperbolic fixed points.

To decide if the bifurcation diagram has a quadratic tangency we first note that $f_{xx} < 0$ for any $\mu > 0$ so that (23) is satisfied. Lastly, using

[8] note that this is not an equation for the "branch" of equilibria $\bar{\mu}$

(24) in (22) we find

$$h(\mu) = f_\mu(x^*(\mu), \mu) = 2 - \frac{1}{\mu^2}$$

and, since $h(\mu_1^*) \neq 0$ and $h(\mu_2^*) \neq 0$, there are quadratic tangencies at the bifurcation points (μ_1^*, x_1^*) and (μ_2^*, x_2^*).

3.3 Near Identity Transformations and Saddle-Nodes

A near identity transformation between a variable x and y is one of the form

$$x = y(1 + \phi(y))$$

where the function ϕ is smooth and $\phi(0) = 0$. It is the latter property which makes the transformation "nearly" the identity when y is small. If one expands the function $\phi(y)$ in a Taylor series about $y = 0$ there would be constants $\phi_n, n = 1, 2, \ldots$ such that

$$x = y(1 + \phi_1 y + \phi_2 y^2 + \phi_3 y^3 + \cdots)$$

Such transformations are often used to simplify expressions in a variety of analytical settings. Here we will demonstrate a use in simplifying the normal form of the saddle node bifurcation of

$$\dot{x} = f(x, \mu)$$

pressumed to occur (without loss of generality) at $(\mu, x) = (0, 0)$. It was previously found that the Taylor series expansion of the differential equation about a quadratic tangency has the form

$$f(x, \mu) = a_0\mu + a_1 x^2 + a_2 x\mu + a_3\mu^2 + O(3) \tag{25}$$

where $a_0 \neq 0$,$a_1 \neq 0$.

We seek to find $\phi_n(\mu), n = 1, 2, \ldots$ in the near identity transformation

$$x = y(1 + \phi_1(\mu)y + \phi_2(\mu)y^2 + \phi_3(\mu)y^3 + \cdots) = y(1 + \phi(y, \mu)) \tag{26}$$

so that the Taylor series expansion for the associated differential equation for $y(t)$ has the form

$$\dot{y} = g(y, \eta) = \eta(\mu) + y^2 + O(3) \tag{27}$$

where η is new parameter which is a function of μ, invertible near $\mu = 0$ and $\eta(0) = 0$. By finding the coefficients $\phi_n(\mu), n = 1, 2, \ldots$ we will have demonstrated all saddle-node bifurcations with quadratic tangency can be transformed into our original model problem (27).

First note that given (26)

$$\dot{x} = (1 + \phi(y, \mu) + y\phi_y(y, \mu))\dot{y} = f(y(1 + \phi(y, \mu)), \mu).$$

Thus

$$\dot{y} = g(y, \eta) = \frac{f(y(1 + \phi(y, \mu)), \mu)}{1 + \phi(y, \mu) + y\phi_y(y, \mu)}. \tag{28}$$

Calculations from here on are laborious [9] but straightforward. One simply uses the expansions (25) and (26) in (28) and then expand about $y = 0$ collecting powers of y along the way. This procedure yields:

$$g(y, \eta) = g_0 + g_1 y + g_2 y^2 + O(y^3)$$

where

$$\begin{aligned} g_0 &= \mu(a_0 + a_3\mu) \\ g_1 &= a_2\mu - 2g_0\phi_1 \\ g_2 &= 2a_1 - 2a_2\phi_1\mu + 8g_0\phi_1^2 - 6g_0\phi_2. \end{aligned}$$

Now we choose ϕ_1 and ϕ_2 so that $g_1 = 0$ and $g_2 = 2a_1$ (which by assumption is nonzero). Then,

$$\begin{aligned} \phi_1(\mu) &= \frac{a_2\mu}{2(a_0 + a_3\mu)} \\ \phi_2(\mu) &= \frac{a_2^2}{6(a_0 + a_3\mu)^2}(2 - \beta). \end{aligned}$$

and

$$\frac{dy}{dt} = g(y, \eta) = g_0 + 2a_1 y^2 + O(y^3).$$

Now we rescale time by setting $\tau = 2|a_1|t$ so that

$$\frac{dy}{d\tau} = \frac{g_0}{2a_1} \pm y^2 + O(y^3)$$

with the $\pm$ being determined by the sign of a_1. Further, we make the identification

$$\eta(\mu) = \frac{g_0}{2a_1} = \frac{\mu(a_0 + a_3\mu)}{2a_1}$$

[9] not too bad when done with a symbollic manipulation program such as Maple or Mathematica

one finds
$$\frac{dy}{d\tau} = \eta(\mu) \pm y^2 + O(y^3) \tag{29}$$
where our new parameter η vanishes at $\mu = 0$ and is invertible in the vicinity of $\mu = 0$. This shows how all saddle-node bifurcations can be transformed into the normal form (29).

3.4 Transcritical Bifurcations

Transcritical bifurcations differ from saddle node bifurcations in that as a parameter is varied fixed points do not appear or disappear. The simplest system having a transcritical bifurcation is:

$$\dot{x} = f(x, \mu) = x(\mu - x)$$

This problem has two branches of fixed points for all $\mu \in \mathbb{R}$:

$$\bar{x}_1(\mu) \equiv 0 \quad , \quad \bar{x}_2(\mu) = \mu$$

It is not hard to determine that $\bar{x}_1$ is stable for $\mu < 0$ and unstable for $\mu > 0$. Conversely, $\bar{x}_2$ is unstable for $\mu < 0$ and stable for $\mu > 0$. Thus, the two branches coalescese and exchange stability at the transcritical bifurcation point $(\mu^*, x^*) = (0, 0)$. More generally one can define transcritical bifurcations by these properties. For instance, if for $\mu \neq \mu^*$ but near μ^* there are two (hyperbolic) fixed points of opposite stability and there is a sole fixed point at $\mu = \mu^*$ then typically the system is said to have a (2 branch) transcritical bifurcation at $(\mu^*, x^*) = (0, 0)$.

Depending on f, a system can have multiple transcritical bifurcations:

Example: Consider

$$\dot{x} = f(x, \mu) = (x - 1)\left((x - 2)^2 - \mu^2)\right)$$

This system has three branches of fixed points

$$\bar{x}_1 = 1 \quad , \quad \bar{x}_2 = 2 + |\mu| \quad , \quad \bar{x}_1 = 2 - |\mu| \quad .$$

which intersect at three different transcritical bifurcation points

$$(\mu_1^*, x_1^*) = (-1, 1)\ ,\ (\mu_2^*, x_2^*) = (1, 1)\ ,\ (\mu_3^*, x_3^*) = (0, 2)\ .$$

Although not necessary, most transcritical bifurcations occur when two branches of fixed points cross *transversely*. By this we mean that local to the bifurcation point the branches form a nonzero angle. A simple example illustrating the importance of this transversality condition is:

Example: Consider

$$\dot{x} = f(x, \mu) = x(\mu - g(x))$$

where $g(x)$ *is a continuously differentiable function with* $g(0) = 0$ *and* $g'(0) \neq 0$*. This system has two branches of fixed points*

$$\bar{x}_1 = 0 \quad , \quad \bar{x}_2 = g(x) \; .$$

These branches cross transversely at a transcritical (TC) point $(\mu^*, x^*) = (0, 0)$ *since* $g'(0) \neq 0$*. Note that if* $g'(0) = 0$ *this might not be the case. For instance, consider* $g(x) = x^2$*. Then the problem would only have a sole fixed point for* $\mu < 0$*.*

Transversality is not a prerequisite for (TC) bifurcations, however. A simple example illustrating this is:

Example: Consider

$$\dot{x} = f(x, \mu) = x^2 - \mu^4$$

This system has two branches of fixed points

$$\bar{x}_1 = \mu^2 \quad , \quad \bar{x}_2 = -\mu^2 \; .$$

These branches do not cross transversely at $(\mu^*, x^*) = (0, 0)$*. The branches are in fact tangent to one another yet* $(0, 0)$ *is a (TC) point since for* $\mu \neq 0$ *there are two fixed point branches of oppositie stabilty that coalescese at* $(0, 0)$*.*

Lastly, we note that some bifurcations might also be called transcritical in that more than two branches coalescese at a sole point. Technically there is no formal naming scheme for such cases but they retain the essential properties of a “2-branch” (TC) bifurcation. Namely, the number of stable and unstable branches of hyperbolic fixed points on either side of μ^* is the same. A simple example is:

Example: Consider

$$\dot{x} = f(x, \mu) = x(x - \mu)(x - 2\mu)(x - 3\mu)$$

This system has four branches of fixed points that intersect at the “degenerate” transcritical bifurcation point $(\mu^*, x^*) = (0, 0)$*.*

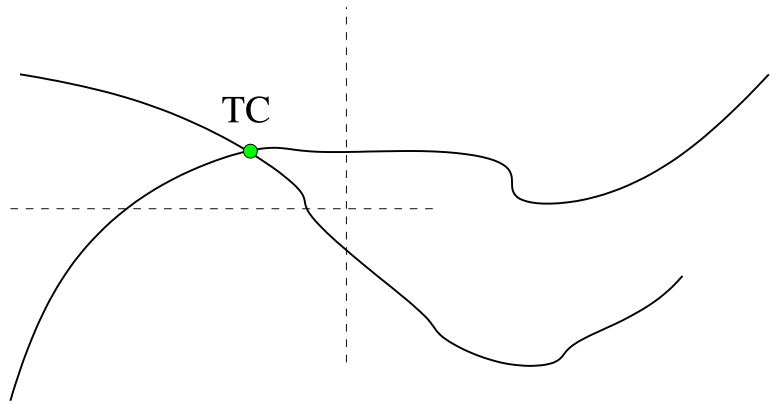

Figure 1: Generic transcritical bifurcation showing that $f(x,\mu) = 0$ does not imply $\mu = \mu(x)$ or $x = x(\mu)$.

3.5 Transcritical Local Theory

Suppose that two disjoint branches of fixed points intersect at a transcritical bifurcation point. From Figure 1 it is easy to see that near (TC), $f(x,\mu) = 0$ does not imply that there is a single function which describes both branches of fixed points.

Now suppose that we consider the Implicit Function Theorem stated below:

Theorem 5 *(IFT) Let* $f : \mathbb{R}^2 \to \mathbb{R}$*,* $f = f(x,\mu)$ *and that* f*,* f_x *and* f_μ *are continuous for all* (x,μ) *near* $(0,0)$*. If*
(i) $f(0,0) = 0$
(ii) $f_\mu(0,0) \neq 0$
then there is a unique function $\bar{\mu}(x)$ *which is continuously differentiable such that*

$$f(x, \bar{\mu}(x)) = 0$$

for all x *near* $x = 0$*.*

Considering this theorem f_μ must vanish at a (TC) bifurcation point. Alternately by swapping x and μ in the theorem, f_x must also vanish. The latter implies that fixed points must (like at saddle node bifurcations) be nonhyperbolic at transcritical bifurcations. Like saddle node bifurcations, there are theorems which place sufficiency conditions on f assuring that transcritical bifurcations exist.

Theorem 6 *Let $f : \mathbb{R}^2 \to \mathbb{R}$, $f = f(x, \mu)$ have continuous derivatives of all orders up to and including degree three, i.e., $f, f_\mu, \ldots f_{xx\mu}$ near (μ^*, x^*). If*

$$\begin{aligned} f(x^*, \mu^*) &= 0 && (30)\\ f_x(x^*, \mu^*) &= 0 && (31)\\ f_\mu(x^*, \mu^*) &= 0 && (32)\\ f_{x\mu}(x^*, \mu^*) &\neq 0 && (33)\\ f_{xx}(x^*, \mu^*) &\neq 0 && (34) \end{aligned}$$

then $\dot{x} = f(x, \mu)$ has a (2 branch) transcritical bifurcation at (μ^*, x^*).

Example: Let

$$\dot{x} = f(x, \mu) = \mu ln(x) + x - 1$$

The first three conditions of the Theorem are:

$$\begin{aligned} f &= \mu ln(x) + x - 1 = 0 && (35)\\ f_x &= \frac{\mu}{x} + 1 = 0 && (36)\\ f_\mu &= ln(x) = 0 && (37) \end{aligned}$$

Solving (36)-(37) one finds $(\mu^*, x^*) = (-1, 1)$ as a candidate for a (TC). It is easily verified that (35) is also satisfied and that

$$\begin{aligned} f_{x\mu} &= \frac{1}{x} \neq 0\\ f_{xx} &= -\frac{\mu}{x^2} \neq 0 \end{aligned}$$

at $(\mu^*, x^*) = (-1, 1)$. By the Theorem one concludes there is a (2-branch) transcritical bifurcation point at $(\mu^*, x^*) = (-1, 1)$.

Much like saddle node bifurcations, such Theorems lead to normal forms for (TC) bifurcations. For example, if (μ^*, x^*) is a pair which satisfies the hypotheses of the Theorem then

$$\dot{x} = \frac{1}{2} f_{\mu\mu}(\mu^*, x^*)(\mu - \mu^*)^2 + f_{x\mu}(\mu^*, x^*)(x - x^*)(\mu - \mu^*) + \frac{1}{2} f_{xx}(\mu^*, x^*)(x - x^*)^2 + O(3)$$

Making the definitions

$$y = x - x^* \quad , \quad \eta = \mu - \mu^*$$

one has the normal form

$$\dot{y} = a\eta^2 + b\eta y + cy^2 + O(3)$$

where a, b, c are constants and $bc \neq 0$.

Example: For the previous example

$$\dot{x} = f(x, \mu) = \mu ln(x) + x - 1$$

where $(\mu^*, x^*) = (-1, 1)$ *is a (TC) bifurcation point,*

$$\dot{x} = (x - 1)(\mu + 1) + \frac{1}{2}(x - 1)^2 + O(3)$$

leading to the normal form

$$\dot{y} = y\left(\eta + \frac{1}{2}y\right) + O(3)$$

3.6 Pitchfork Bifurcations

The generic pitchfork bifurcation (PF) can be described by the system

$$\dot{x} = f(x, \mu) = \mu x - x^3 = x(\mu - x^2)$$

whose loci of fixed points can be described by

$$\begin{aligned} \bar{x} &= 0 \quad , \quad \forall \mu \\ \bar{\mu}(x) &= x^2 \quad , \quad \forall x \, . \end{aligned}$$

It is not hard to show that $\bar{x} = 0$ is stable only for $\mu < 0$ and unstable otherwise. Also, both fixed points on the branch $\bar{\mu} = x^2$ are stable resulting in a characteristic "pitchfork" shape illustrated in Figure 2. This system is said to have a pitchfork bifurcation at $(\mu^*, x^*) = (0, 0)$. Since two stable fixed points surround and unstable (hyperbolic) fixed point near $(0, 0)$ the pitchfork is said to be supercritical.

The stability of the branches in this example can be reversed by making the transformation $t \to -t$ resulting in

$$\dot{x} = f(x, \mu) = -\mu x + x^3 = -x(\mu - x^2)$$

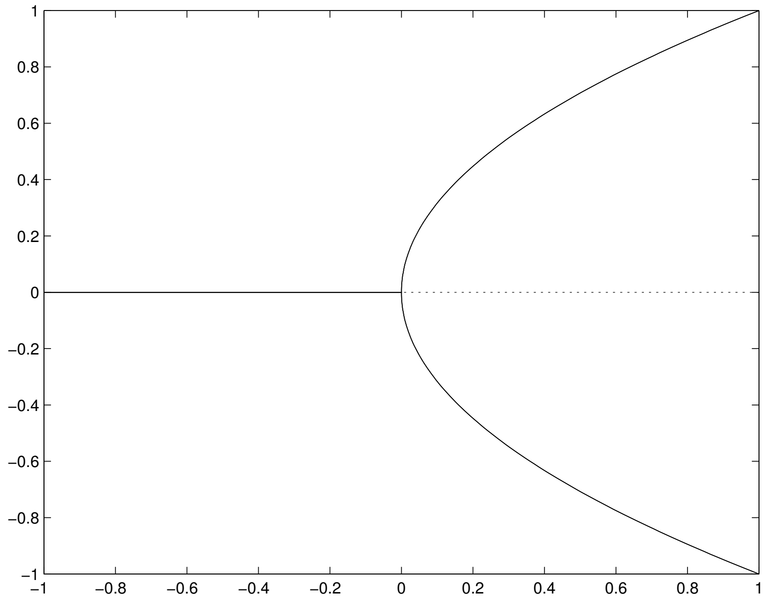

Figure 2: Generic pitchfork bifurcation for $\dot{x} = \mu x - x^3$

Then, the branch $\bar{\mu}(x)$ is unstable and near the bifurcation point two unstable fixed points surround a stable fixed point. Such a pitchfork is said to be subcritical.

Example: Any system of the form

$$\dot{x} = x(\mu - g(x))$$

where $g(0) = g'(0) = 0$ *and* $g''(0) \neq 0$ *will have a pitchfork bifurcation at* $(\mu^*, x^*) = (0, 0)$*. The criticality (super or sub) of the pitchfork will depend on the sign of* $g''(0)$*. These conditions on* $g(x)$ *imply that* g *is locally quadratic near* $x = 0$*. For this reason the system may be said to have a pitchfork bifurcation of quadratic tangency at* (μ^*, x^*)*. Thus, both of the following systems have pitchfork bifurcations of quadratic tangency at* $(0, 0)$*.*

$$\begin{aligned} \dot{x} &= x(\mu - e^{-x^2}) \\ \dot{x} &= x(\mu - sin(x^2)) \end{aligned}$$

Some pitchfork bifurcations clearly do not have quadratic tangencies. For instance, consider

$$\dot{x} = x(\mu - g(x)) = x(\mu - x^4)$$

Here $g(0) = g'(0) = g''(0) = 0$.

Example: Consider

$$\dot{x} = f(x, \mu) = -x(x^2 + \mu^2 - 1)$$

The equilibria lie on $x = 0$ and the circle $x^2 + \mu^2 = 1$ so this system clearly has two pitchfork bifurcations at $(\mu^*, x^*) = (-1, 0)$ and $(\mu^*, x^*) = (1, 0)$. Since $x^2 + \mu^2 - 1 < 0$ only inside the circle, $\bar{x} = 0$ is unstable only for $\mu \in [-1, 1]$ so that both pitchfork bifurcations are supercritical.

Lastly, like saddle node and transcritical bifurcations there are theorems which can be used to determine whether a system has a pitchfork bifurcation. In the next section we iclude such a theorem for detecting pitchfork bifurcations having quadratic tangencies. Here, we that theorem in the following example.

Example:

3.7 Theorems for Saddle-Node, Transcritical and Pitchfork Bifurcations

Here we summarize some theorems for "generic" saddle-node, transcritical and pitchfork bifurcations of

$$\dot{x} = f(x, \mu) \quad , \quad x, \mu \in \mathbb{R}.$$

Proofs use the implicit function theorem, near identity transformations and various other transformations. In all of the the theorems below we assume that f has continuous (mixed) derivatives up to third order (fourth for pitchforks) for all (μ, x) near the bifurcation point (μ^*, x^*).

Theorem 7 *If there is a pair (μ^*, x^*) for which*

$$\begin{aligned} f(x^*, \mu^*) &= 0 \quad (38) \\ f_x(x^*, \mu^*) &= 0 \quad (39) \\ f_\mu(x^*, \mu^*) &\neq 0 \quad (40) \\ f_{xx}(x^*, \mu^*) &\neq 0 \quad (41) \end{aligned}$$

then $\dot{x} = f(x, \mu)$ has a saddle-node bifurcation with quadratic tangency at (μ^*, x^*).

Transcritical (2-branch)

Theorem 8 *If there is a pair (μ^*, x^*) for which*

$$f(x^*, \mu^*) = 0 \tag{42}$$
$$f_x(x^*, \mu^*) = 0 \tag{43}$$
$$f_\mu(x^*, \mu^*) = 0 \tag{44}$$
$$f_{x\mu}(x^*, \mu^*) \neq 0 \tag{45}$$
$$f_{xx}(x^*, \mu^*) \neq 0 \tag{46}$$

then $\dot{x} = f(x, \mu)$ has a (2 branch) transcritical bifurcation at (μ^*, x^*).

Pitchfork (Quadratic Tangency)

Theorem 9 *If there is a pair (μ^*, x^*) for which*

$$f(x^*, \mu^*) = 0 \tag{47}$$
$$f_x(x^*, \mu^*) = 0 \tag{48}$$
$$f_\mu(x^*, \mu^*) = 0 \tag{49}$$
$$f_{xx}(x^*, \mu^*) = 0 \tag{50}$$
$$f_{x\mu}(x^*, \mu^*) \neq 0 \tag{51}$$
$$f_{xxx}(x^*, \mu^*) \neq 0 \tag{52}$$

then $\dot{x} = f(x, \mu)$ has a pitchfork bifurcation with quadratic tangency at (μ^*, x^*).

3.8 Bifurcation diagrams for one dimensional equations

We have been studying the stability structure of

$$\dot{x} = f(x, \mu) \quad , x, \mu \in \mathbb{R}$$

over ranges of μ. Assuming a certain degree of smoothness of f in (x, μ) we stated three theorems concerning the existence of saddle-node (quadratic tangency), transcritical (two-branch) and pitchfork (quadratic tangency) bifurcations. These theorems can be used to determine exactly what the bifurcation diagrams look like near the bifurcation point (μ^*, x^*).

Several other "bad" things can happen.

Definition 8 *Let*

$$\dot{x} = f(x, \mu) \quad , \quad x(t) \in \mathbb{R},$$

and suppose that $f(\bar{x}) = 0$. Further suppose that there is an $\epsilon > 0$ such that on the interval $(\bar{x} - \epsilon, \bar{x} + \epsilon)$ f has no other roots. Then we say that $\bar{x}$ is an isolated fixed point.

Not all fixed points are isolated. We show this by way of example. First, let us define a special function:

$$\phi(\mu) = \begin{cases} \mu^2 & \mu \leq 0 \\ 0 & \mu > 0 \end{cases} .$$

This function has a few properties. First $\phi > 0$ if $\mu < 0$. Also, $\phi = 0$ for all $\mu \geq 0$. Moreover, $\phi(\mu)$ and $\phi'(\mu)$ are continuous on $\mathbb{R}$.

Now, consider the problem

$$\dot{x} = f(x, \mu) = \phi(\mu)x.$$

Here f, f_x and f_μ are continuous on $\mathbb{R}^2$ and in particular at $\mu = 0$. However, note that for each fixed $\mu \geq 0$, the problem has an infinite number of fixed points. For example, $f(x, 1) = 0$ for any x. Clearly, such fixed points are not isolated by the definition above. Moreover, given our previous definitions of stability and asymptotic stability, these non-isolated fixed points are stable but not asymptotically stable. We can see this by way of example too. For instance, set $\mu = 1$. Then the associated initial value problem is trivial:

$$\frac{dx}{dt} = 0 \quad , \quad x(0) = x_0.$$

and has the solution $x(t) = x_0$ for all t. Certainly $\bar{x} = 1$ is a non-isolated fixed point but no matter how close x_0 is to $\bar{x}$, $x(t)$ does not approach $\bar{x}$ as $t \to \infty$. Therefore, $\bar{x} = 1$ is not asymptotically stable. On the other hand, for every x_0 close to $\bar{x} = 1$, $x(t)$ does not move away. Hence it is stable.

Even if fixed points are isolated, the function $\phi(\mu)$ can be used to create examples of relatively smooth functions f which yield unusual bifurcation diagrams. Toward this end, let us first define two new functions:

$$\phi_+(\mu) = \phi(\mu) + \phi(1 - \mu) \tag{53}$$

$$\phi_-(\mu) = \phi(\mu) - \phi(1 - \mu) \tag{54}$$

Graphs of these functions show that both are zero for $\mu \in [0, 1]$. For $\mu \not\in [0, 1]$, $\phi_+ > 0$. But, ϕ_- is positive only for $\mu < 0$. Otherwise, it is negative or zero.

Now, consider the following problems:

A) $\dot{x} = f(x, \mu) \equiv (x^2 - \mu^2)^2$

B) $\dot{x} = f(x, \mu) \equiv (x^4 - \phi(\mu))^2$

C) $\dot{x} = f(x, \mu) \equiv (x^4 - \phi_+(\mu))^2$

D) $\dot{x} = f(x, \mu) \equiv (x^4 - \phi_-(\mu))^2$

E) $\dot{x} = f(x, \mu) \equiv \phi_+(\mu) - x^4$

F) $\dot{x} = f(x, \mu) \equiv \mu^2(4x^2 - 1)^2$

In Figure 3, the bifurcation diagrams are drawn for each example. Notice that, in all but Figure 3E, two unstable fixed points coexist at the same μ values. For example, in Figure 3B the branches $\bar{x} = \bar{x}_+(\mu) = \sqrt{|\mu|}$ and $\bar{x} = \bar{x}_-(\mu) = -\sqrt{|\mu|}$ are both unstable for $\mu < 0$. This is an example where branch stability does not alternate from branch to branch for fixed μ. The reason this can happen is due to the fact that the branches are nonhyperbolic. That is to say, at each fixed μ the fixed points $\bar{x}_\pm(\mu)$ are non hyperbolic. A simple calculation can be used to verify this. For example B),

$$f_x(x, \mu) = 8x^3(x^4 - \phi(\mu)).$$

Then,

$$f_x(\bar{x}_\pm(\mu), \mu) = 0$$

for all $\mu < 0$. Moreover, since $f_x(0, \mu) = 0$ too, <u>all</u> fixed points are nonhyperbolic!

Now we state a theorem:

Theorem 10 *Let*

$$\dot{x} = f(x, \mu) \quad , \quad x(t) \in \mathbb{R} \tag{55}$$

where $f(x, \mu)$ and $f_x(x, \mu)$ are continuous on $\mathbb{R}$. If at some fixed μ, (55) has exactly n isolated <u>hyperbolic</u> fixed points $\bar{x}_k, k = 1, 2, \ldots n$ in some interval then the stability of the fixed points must alternate.

This theorem is not violated by our previous examples because (in all but example E) all the fixed points are isolated but nonhyperbolic. This leads us to a strategy for creating bifurcation diagrams:

i) Locate all fixed points by find all pairs (x, μ) that satisfy $f = 0$.

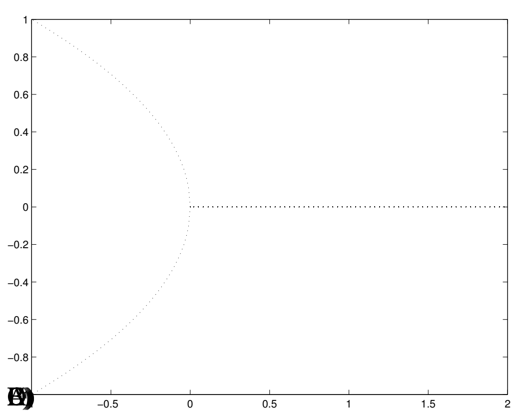

Figure 3: Figure shows bifurcation diagrams for examples **A)-F)** The horizontal axis is μ, the vertical is x. Dashed lines indicate unstable fixed points. Solid lines indicate stable fixed points

ii) Locate all the nonhyperbolic fixed points (x_{nh}, μ_{nh}) by finding all solutions (x, μ) of the system:

$$\begin{aligned} f(x,\mu) &= 0 \\ f_x(x,\mu) &= 0 \end{aligned}$$

iii) Compute the stability of a single branch of fixed points (pick the easiest to deal with).

iv) Hopefully, all fixed points will be isolated and you have a finite number of nonhyperbolic fixed points. Then, by using the theorem you can deduce the stability of the other fixed points since they must all be isolated and hyperbolic.

Recipes such as this can be dangerous. Here's a word of warning. Suppose you decided to locate all transcritical bifurcations by finding (x, μ) pairs which are solutions to the three equations

$$f(x,\mu) = 0 \tag{56}$$

$$f_x(x,\mu) = 0 \tag{57}$$

$$f_\mu(x,\mu) = 0 \tag{58}$$

It might turn out that it is easiest to solve (57)-(58) and the verify that the values of (x, μ) so obtained also satisfy (56). Thus, it is a nonhyperbolic fixed point. However, it may not be the only hyperbolic fixed point. For example, at a saddle-node bifurcation the fixed point is nonhyperbolic but $f_\mu \neq 0$!

3.9 Structural Stability and Bifurcations

Suppose we wish to exam the bifurcation structure of the equation:

$$\dot{x} = f(x,\lambda) \quad , \quad \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{R}^k$$

We have examined saddle node, transcritical and pitchfork bifurcations for the case $k = 1$. When a second parameter is introduced some of these bifurcation diagrams are radically altered. Consider the following generic problems where $\lambda = (\lambda_1, \lambda_2)$:

(SN) $\quad \dot{x} = f(x,\lambda) = \lambda_1 + \lambda_2 + x^2$

(TC) $\quad \dot{x} = f(x,\lambda) = \lambda_2 + \lambda_1 x - x^2$

(PF) $\dot{x} = f(x, \lambda) = \lambda_2 + \lambda_1 x - x^3$

When $\lambda_2 = 0$, (SN), (TC) and (PF) have saddle node, transcritical and pitchfork bifurcations $(x, \lambda_1) = (0, 0)$, respectively, as the parameter λ_1 is varied. One can show (as was done in class) that if $\lambda_2 \neq 0$ is fixed that these bifurcations in λ_1 are "preserved" only for the saddle node case. To describe in a mathematically rigorous manner what "preserve" means we need to define a few concepts.

Definition 9 *If $h : \mathbb{R} \to \mathbb{R}$ is continuous and invertible on $\mathbb{R}$ we say that h is a homeomorphism on $\mathbb{R}$. If in addition, h is continuously differentiable on $\mathbb{R}$ we say that h is a diffeomorphism on $\mathbb{R}$.*

For example, $h(x) = 2x + 1$, $h(x) = tanh(x)$ and $h(x) = x^3$ are all diffeomorphisms. Certain piecewise defined functions are homeomorphisms but not diffeomorphisms. For example,

$$h(x) = \begin{cases} 2x & x \leq 0 \\ x & x > 0 \end{cases}$$

is a homeomorphism on $\mathbb{R}$ but not a diffeomorphism on $\mathbb{R}$ since $h'(0)$ is undefined.

One use of diffeomorphisms [10] is to simplify differential equations. For example, suppose $h(x)$ is a diffeomorphism and we let $y(t) = h(x(t))$. Then if $x(t)$ is a solution of

$$\dot{x} = f(x) \quad ,$$

$y(t)$ must be a solution of

$$\dot{y} = F(y) = h'(h^{-1}(y))f(h^{-1}(y))$$

If one chooses h in an intelligent way the resulting function F may be simpler than the original $f(x)$. Solutions to the original problem can be recovered easily since if one knows $y(t)$ then $x(t) = h^{-1}(y(t))$, i.e., the inverse exists. Moreover, all of the flow "topology" is retained. For instance, if the y equation has a finite number of isolated fixed points, so will the x equation.

Definition 10 *Two scalar differential equations $\dot{x} = f(x)$ and $\dot{y} = F(y)$ each with a finite number of isolated fixed points are said to be*

[10] of which there are many uses

topologically equivalent if there exists a homeomorphism $h(z)$ on $\mathbb{R}$ such that

a) if for every solution $x(t)$ of $\dot{x} = f(x)$, $y(t) = h(x(t))$ is a solution of $\dot{y} = F(y)$,
b) if a fixed point of $\bar{x}$ of $\dot{x} = f(x)$ is stable (unstable) then $\bar{y} = h(\bar{x})$ is a stable (unstable) fixed point of $\dot{y} = F(y)$.

This defintion basically says that all the essential stability information is retained but the exact location of fixed points and the rate of approach to stable fixed points may be altered. Now for:

Definition 11 *Let $\dot{x} = f(x, \lambda)$ where $\lambda = (\lambda_1, \ldots \lambda_k)$ is a parameter in $\mathbb{R}^k$. The equation $\dot{x} = f(x, \bar{\lambda})$ is structurally stable if there exists an $\epsilon > 0$ such that $\dot{x} = f(x, \bar{\lambda})$ is is topologically equivalent to $\dot{x} = f(x, \lambda)$ for all λ with $\parallel \lambda - \bar{\lambda} \parallel < \epsilon$.*

The norm $\parallel \cdot \parallel$ can be any norm but usually taken to be the Euclidean distance norm, i.e., for $k = 2$ the distance:

$$\parallel \lambda - \bar{\lambda} \parallel = \sqrt{(\lambda_1 - \bar{\lambda}_1)^2 + (\lambda_2 - \bar{\lambda}_2)^2}.$$

In our in class examples, it is clear that only SN is structurally stable at $\bar{\lambda} = (0, 0)$. Both the TC and PF bifurcations are structurally unstable.

4 Dynamics on S^1

For the problem

$$\dot{x} = f(x) \quad , \quad f : \mathbb{R} \to \mathbb{R}$$

the function f can be consider as a vector field on $\mathbb{R}$. As such, solutions $x(t)$ form a "flow" on $\mathbb{R}$ and one can discuss the dynamics on $\mathbb{R}$ induced by the vector field f. Sometimes such dynamics will involve a discussion of fixed-point stability and bifurcations. Other times f never vanishes and $x(t)$ is monotonic in t.

Differential equations can induce flows on other "geometrical" objects. One such "object" is S^1. S^1 is defined to be the unit circle (centered at the origin) in $\mathbb{R}^2$. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $f(\theta)$ is 2π-periodic in θ, i.e.,

$$f(\theta + 2\pi) = f(\theta) \quad , \quad \forall\theta.$$

Consider a point $P(t)$ with coordinates $P(t) = (x(t), y(t))$ constrained to move on S^1. If we define the polar coordinate transformation $(x, y) \leftrightarrow (r, \theta)$ via

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

then the motion of P can unambiguously be described by the set of differential equations:

$$\begin{aligned} \dot{r} &= 0 \\ \dot{\theta} &= f(\theta) \end{aligned}$$

Here the periodicity of f is essential to "unambigously" describe the dynamics of P on S^1 in the sense that we <u>require</u> the velocity to be the same at every point on S^1. Note that this is different than the motion of a particle moving on the circle whose speed increases upon every revolution. Such a particle motion might better be described by the non-autonomous equation

$$\dot{\theta} = f(\theta, t)$$

We <u>define</u> dynamics on S^1 to be that flow induced by a periodic $f(\theta)$.

Given this definition, then any flow on $\mathbb{R}$ induced by a T-periodic function $F(x)$ can be mapped onto a flow on S^1. To see this suppose

$$\dot{x} = F(x) \quad , \quad F : \mathbb{R} \to \mathbb{R} \quad , \quad F(x+T) = F(x) \quad \forall x \in \mathbb{R}.$$

Now define

$$\theta = \frac{2\pi x}{T}.$$

Then

$$\dot{\theta} = f(\theta) \equiv \frac{2\pi}{T} F\left(\frac{T\theta}{2\pi}\right). \tag{59}$$

Note that the calculations

$$\begin{aligned} f(\theta + 2\pi) &= \frac{2\pi}{T} F\left(\frac{T}{2\pi}(\theta + 2\pi)\right) \\ &= \frac{2\pi}{T} F\left(\frac{T\theta}{2\pi} + T\right) \\ &= f(\theta) \end{aligned}$$

verify that f is 2π-periodic in θ so that (59) does describe a flow on S^1.

Example: $\dot{x} = \sin(2\pi x)$ *is a* 1*-periodic flow on* $\mathbb{R}$ *which can be mapped to the flow* $\dot{\theta} = sin(\theta)$ *on* S^1.

Even if the vector field $F(x)$ on $\mathbb{R}$ is not periodic in x, the flow can be mapped to S^1 using a transformation which compactifies the real line onto $(-\pi, \pi)$. For example, let

$$\dot{x} = F(x) \quad , \quad F : \mathbb{R} \to \mathbb{R} \quad .$$

Choose any function $\phi : \mathbb{R} \to \mathbb{R}$ that is smooth, invertible and has the range $(-\pi, \pi)$. An example of such a function is

$$\phi(x) = \pi \tanh x.$$

Then, let

$$\theta(t) = \phi(x(t))$$

so that, for $\psi(x) = \phi^{-1}(x)$,

$$x(t) = \psi(\theta(t)).$$

Differentiation of the latter expression in t yields:

$$\dot{x} = \psi'(\theta)\dot{\theta} = F(\psi(\theta))$$

or that

$$\dot{\theta} = f(\theta) \equiv \frac{F(\psi(\theta))}{\psi'(\theta)} \tag{60}$$

Here, since ϕ is invertible on $\mathbb{R}$, $\psi'(\theta) > 0$ strictly. Moreover, if $\bar{x}$ is a fixed point of $\dot{x} = F(x)$ then, from (60), $\bar{\theta} = \phi(\bar{x})$ is a fixed point of $\dot{\theta} = f(\theta)$.

Below is an example of this compactification process.

Example: Let $x(t)$ be a solution of $\dot{x} = x^2 + 1$ and define

$$\theta = \pi \ tanh(x)$$

Note that $\theta \in (-\pi, \pi)$ so that the dynamics of $\theta(t)$ can be mapped into S^1. By differentiating the above expression in t,

$$\dot{\theta} = \pi sech^2 x \ \ (x^2 + 1)$$

which by using the inverse transformation $x = tanh^{-1}\left(\frac{\theta}{\pi}\right)$ and the hyperbolic identity $sech^2\, x = 1 - tanh^2\, x$ becomes

$$\dot{\theta} = F(\theta)$$

where

$$F(\theta) = \pi \left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 + tanh^{-1}\left(\frac{\theta}{\pi}\right)\right).$$

The function F is defined only on the open interval $(-\pi, \pi)$. From a graph it can be seen that $F > 0$ but that $F \to 0$ as $|\theta| \to \pi^-$.

To extend the dynamics of $\theta(t)$ to be defined on all of S^1 we let

$$\bar{F}(\theta) = \begin{cases} F(\theta) & \theta \in (-\pi, \pi) \\ 0 & \theta = \pi \end{cases}$$

and set

$$\dot{\theta} = \bar{F}(\theta)$$

By doing so, the fixed point at $\theta = \pi$ can be regarded as the "point" $x = \infty$ in

$$\dot{x} = x^2 + 1$$

Specifically, the limit $\theta \to -\pi^+$ corresponds to $x \to -\infty$ whereas $\theta \to \pi^-$ corresponds to $x \to \infty$. With this interpretation the equation $\dot{\theta} = \bar{F}(\theta)$ can be regarded as a mapping of the original problem on the extended reals $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ to S^1. Moreover, since $\bar{F} > 0$ on $(-\pi, \pi)$, $\theta(t)$ increases toward the fixed point $\bar{\theta} = \pi$ (despite the fact it is unstable, i.e. "half-stable"). This reflects the dynamics of the "blowup" in the original problem for x. It does not, however, say anything about how much time this approach to "∞" at $\bar{\theta} = \pi$ will take. From the exact solution $x(t) = \tan(t + t_0)$ we know it takes a finite time.

4.1 Periodic Solutions on S^1

From previous analyses we know that for smooth f, there do not exist (nontrivial) T-periodic solutions $\theta(t)$ to

$$\dot{\theta} = f(\theta) \tag{61}$$

even if f is periodic in its argument. For instance, solutions $\theta(t)$ of

$$\dot{\theta} = sin(\theta)$$

are not periodic in t even though $f(\theta)$ is periodic in θ.

Neverthless, the dynamics that equation (61) imposes on S^1 can be periodic. For example, suppose that f is 2π-periodic and strictly positive, i.e., $f(\theta) > 0, \forall\theta \in [0, 2\pi)$. The coordinate of a point $P(t)$ on S^1 at time t is

$$P(t) = (x(t), y(t)) = (cos\ \theta(t)\ , sin\ \theta(t)).$$

Since $f > 0$, $\theta(t)$ increases in t indefinitely so that after a certain time $T > 0$ the point $P(t)$ returns to its original position, i.e.,

$$P(t + T) = P(t).$$

In fact since f is periodic this period of revolution T is the same the next time around and

$$P(nT + t) = P(t) \quad , \quad n = 1, 2, 3, \ldots.$$

From the differential equation the period T can easily be deduced as:

$$T = \int_0^{2\pi} \frac{d\theta}{f(\theta)}$$

Example: The equation $\dot{\theta} = f(\theta) = \omega$ where $\omega > 0$ is constant describes pure rotation on S^1. The solution with $\theta(0) = \theta_0$ is $\theta(t) = \omega t + \theta_0$ and is not periodic in t even though $f(\theta) = \omega$ is periodic in θ (every constant function is). The period is clearly $T = 2\pi/\omega$ as can be computed:

$$T = \int_0^{2\pi} \frac{d\theta}{\omega} = \frac{2\pi}{\omega}$$

Example: For $\omega > a > 0$ the equation

$$\dot{\theta} = \omega - a \sin\ \theta$$

describes a nonlinear oscillation on S^1 since $f(\theta) > 0$. The period T can be computed using tables and careful evaluations of necessary limits:

$$T = \int_{-\pi}^{\pi} \frac{d\theta}{\omega - a \sin\ \theta} = \frac{2\pi}{\sqrt{\omega^2 - a^2}}$$

In particular, using the indefinite integral

$$F(x) \equiv \int^x \frac{d\theta}{\omega - a \sin\ \theta} = \frac{1}{2\sqrt{\omega^2 - a^2}} arctan \left(\frac{\omega \tan(x/2) - a}{\sqrt{\omega^2 - a^2}} \right)$$

one finds

$$T = \lim_{\epsilon \to 0^+} F(\pi - \epsilon) - F(-\pi + \epsilon) = \frac{2\pi}{\sqrt{\omega^2 - a^2}}.$$

For a near ω the problem is close to a saddle-node bifurcation and the period increases greatly. In this instance $\bar{\theta} = \frac{\pi}{2}$ acts as a "ghost" fixed point and the solution takes a long time to traverse by that value - making T large. This large value can be estimated as follows:

$$T = \frac{2\pi}{\sqrt{(\omega - a)(\omega + a)}} \sim T_B \equiv \frac{2\pi}{\sqrt{2a(\omega - a)}}$$

as $a \to \omega$. Here the symbol $\sim$ means "asymptotic to" and has a precise meaning which we will not define at the moment.

4.2 Saddle-Nodes, Ghosts and Bottleneck durations

In the preceding example

$$\dot{\theta} = f(\theta) = \omega - a \sin\ \theta$$

the equation undergoes a saddle-node bifurcation at $\omega = a$ as the parameter a is varied. The fixed point at this bifurcation is $\bar{\theta} = \frac{\pi}{2}$. For $a > \omega$ the problem has two fixed points, one of which is stable and the other unstable. But, for $a < \omega$ an oscillation persists. This is true even if $0 < \omega - a \ll 1$ but then the period T is very large. This is due to the fact that $f(\theta)$ is very small near $\theta = \frac{\pi}{2}$. In this instance, the flow has a "bottleneck" due to the "ghost" fixed point $\bar{\theta} = \frac{\pi}{2}$. Here, $\bar{\theta} = \frac{\pi}{2}$ is called a "ghost" fixed point because even though it technically is not a fixed point, it acts like one by slowing the flow down for a long time.

The same "bottleneck" occurs when any first order equation (be it on $\mathbb{R}$ or S^1) is near a saddle-node bifurcation. Here we will derive an asymptotic approximation of the period T for the case when the equation is near a saddle-node of quadratic tangency. Toward this end, let $\theta(t)$ be a solution of

$$\dot{\theta} = f(\theta)$$

where f is strictly positive, 2π-periodic and $\bar{\theta}$ is a local minimum of f with

$$f'(\bar{\theta}) = 0 \quad , \quad f''(\bar{\theta}) > 0.$$

At a saddle-node bifurcation of quadratic tangency these conditions are satisfied.

Now define the small positive parameter

$$\epsilon = f(\bar{\theta}) > 0$$

and Taylor expand $f(\theta)$ about $\bar{\theta}$ as follows:

$$\begin{aligned} f(\theta) &= f(\bar{\theta}) + f(\bar{\theta})(\theta - \bar{\theta}) + \frac{1}{2!}f''(\bar{\theta})(\theta - \bar{\theta})^2 + g(x) \\ &= \epsilon + \frac{1}{2!}f''(\bar{\theta})(\theta - \bar{\theta})^2 + g(x) \end{aligned}$$

where we have defined x as the variance from $\bar{\theta}$ as

$$x = \theta - \bar{\theta}$$

and $g(x)$ is the exact remainder. By solving for g as follows:

$$g(x) = f(\theta) - \epsilon - \frac{1}{2!}f''(\bar{\theta})(\theta - \bar{\theta})^2$$

it can be verified that

$$g(0) = g'(0) = g''(0) = 0. \tag{62}$$

The exact period of the oscillation is:

$$T = \int_0^{2\pi} \frac{d\theta}{\epsilon + \frac{1}{2!}f''(\bar{\theta})(\theta - \bar{\theta})^2 + g(x)}.$$

Since the integrand is 2π-periodic in θ this integral can be written

$$T = \int_{-\pi}^{\pi} \frac{dx}{\epsilon + bx^2 + g(x)} \tag{63}$$

where we have defined

$$b = \frac{1}{2!} f''(\bar{\theta}) > 0.$$

Now, let

$$y = \frac{x}{\sqrt{\epsilon}}$$

so that (63) becomes

$$T = \frac{1}{\sqrt{\epsilon}} \int_{-\frac{\pi}{\sqrt{\epsilon}}}^{\frac{\pi}{\sqrt{\epsilon}}} \frac{dy}{1 + by^2 + \frac{g(\sqrt{\epsilon}y)}{\epsilon}} \tag{64}$$

If the term $\frac{g(\sqrt{\epsilon}y)}{\epsilon}$ in the integrand is sufficiently small as to not contribute significantly to the integral, then it should be the case that

$$T \sim \frac{1}{\sqrt{\epsilon}} \int_{-\frac{\pi}{\sqrt{\epsilon}}}^{\frac{\pi}{\sqrt{\epsilon}}} \frac{dy}{1 + by^2}$$

or, in the limit as $\epsilon \to 0$,

$$T \sim \frac{1}{\sqrt{\epsilon}} \int_{-\infty}^{\infty} \frac{dy}{1 + by^2}$$

By evaluating the latter integral and using the definition of b and ϵ we arrive at

$$T \sim T_B \equiv \pi \sqrt{\frac{2}{f(\bar{\theta}) f''(\bar{\theta})}} \tag{65}$$

This approximation is valid for a general bottleneck duration near a saddle-node bifurcation of *quadratic* tangency. For example, had the saddle-node been "quartic" and had a Taylor series like $f = \epsilon + bx^4 + O(x^5)$ the approximation would be different.

Also, the argument made to arrive at this approximation should have made the reader a bit suspicious since the term $\frac{g(\sqrt{\epsilon}y)}{\epsilon}$ looks like it might be large, especially since there is a division by ϵ (which is very small). However, the remainder $g(x)$ in the Taylor series has the form

$$g(x) = \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + O(x^5)$$

so that

$$\frac{g\left(\sqrt{\epsilon}y\right)}{\epsilon} = \frac{1}{3!}\sqrt{\epsilon}y^3 + \frac{1}{4!}\epsilon y^4 + O(\epsilon^{3/2}).$$

These informal calculations show that $\frac{g\left(\sqrt{\epsilon}y\right)}{\epsilon} = O(\sqrt{\epsilon})$ is indeed small in the limit $\epsilon \to 0$. A very careful treatment indicating the accuracy of the approximation (65) can be made but we do not include it here. The basic idea of such proofs is to show that the difference between the exact value T and the asymptotic approximation T_B is of a lower order. Instead of discussing these issues we illustrate an application of the approximation.

Example: We show how to use (65) to obtain an estimate for the (large) period of the oscillation on S^1 induced by

$$\dot{\theta} = f(\theta) = \omega - a\sin\,\theta$$

when $a \simeq \omega$. The local mimina of f occurs at $\theta = \frac{\pi}{2}$ at which $f'(\theta)$ vanishes and

$$\begin{aligned} f\left(\frac{\pi}{2}\right) &= \omega - a \\ f''\left(\frac{\pi}{2}\right) &= a \end{aligned}$$

From (65),

$$T \sim T_B = \pi\sqrt{\frac{2}{a(\omega - a)}}.$$

This is the same result we obtained earlier for this problem.

5 Planar Systems - Preliminary Definitions

By a planar system of differential equations we mean a system of the form:

$$\begin{aligned}\dot{x_1} &= f_1(x_1, x_2) \\ \dot{x_2} &= f_2(x_1, x_2)\end{aligned}$$

where $f_i : \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$. This system can be written in the compact form

$$\dot{x} = f(x) \tag{66}$$

by making the identifications:

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \quad , \quad f(x) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} \quad .$$

In particular, f is a vector-valued function, i.e., $f : \mathbb{R}^2 \to \mathbb{R}^2$. Also, we may write the column vector x above as $x = (x_1, x_2)^T$ where the superscript ^{T} means transpose. Unless otherwise stated we will assume that solutions exist for all time and that the components of f are twice continuously differentiable on $\mathbb{R}^2$.

To make latter definitions more compact we define the Euclidean norm of $x \in \mathbb{R}^2$ by

$$\| x \| = \sqrt{x_1^2 + x_2^2} \quad , \quad x = (x_1, x_2)^T \quad .$$

Then the Euclidean distance between $x, y \in \mathbb{R}^2$ is

$$d(x, y) = \| x - y \| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

We shall then define a δ-neighbourhood of x as the set of points a distance at most δ from x, or:

$$N_\delta(x) = \{y \in \mathbb{R}^2 \ : \| x - y \| < \delta.\}$$

Definition 12 *$\bar{x}$ is a <u>fixed point</u> of $\dot{x} = f(x)$, $f : \mathbb{R}^2 \to \mathbb{R}^2$, if $f(\bar{x}) = 0$.*

Definition 13 *A fixed point $\bar{x}$ of $\dot{x} = f(x)$ is <u>isolated</u> if $\exists \delta > 0$ such that $y \in N_\delta(\bar{x})$ and $y \neq x$ implies $f(y) \neq 0$.*

Thus, if $\bar{x}$ is an isolated fixed point, there is some (small) neighbourhood of $\bar{x}$ that contains no other fixed point. We will adopt the following definitions related to the stability of fixed points.

Definition 14 *A fixed point $\bar{x}$ of $\dot{x} = f(x)$ is Liapunov stable if $\forall \epsilon > 0$ $\exists \delta > 0$ such that if $x(t)$ is a solution of*

$$\dot{x} = f(x) \quad , \quad x(t_0) = x_0 \in N_\delta(\bar{x})$$

then

$$\| x(t) - \bar{x} \| < \epsilon \quad , \quad \forall t \geq t_0 \, .$$

Notice that for this definition to make sense $\delta \leq \epsilon$ else there would be some initial conditions $x_0 \in N_\delta(\bar{x})$ for which $x(t)$ would initially be outside the neighbourhood $N_\epsilon(\bar{x})$ it is required to remain in for all $t \geq t_0$. In words, this definition implies that the solutions $x(t)$ remain close to the fixed point for all time if the initial condition is sufficiently close to $\bar{x}$.

Definition 15 *A fixed point $\bar{x}$ of $\dot{x} = f(x)$ is stable if it is Liapunov stable.*

Thus, for our conventions, stable and Liapunov stable are equivalent.

Definition 16 *A fixed point $\bar{x}$ of $\dot{x} = f(x)$ is attracting if $\exists \delta > 0$ such that if $x(t)$ is a solution of*

$$\dot{x} = f(x) \quad , \quad x(t_0) = x_0 \in N_\delta(\bar{x})$$

then

$$\lim_{t \to \infty} \| x(t) - \bar{x} \| = 0$$

Notice that this definition does not preclude the possibility of $x(t)$ leaving the neighbourhood $N_\delta(\bar{x})$ for some time. However, it does imply that $x(t)$ must eventually return to and stay in $N_\delta(\bar{x})$. Since large excursions are possible, some attracting fixed points are normally not thought of as "stable".

Definition 17 *A fixed point $\bar{x}$ of $\dot{x} = f(x)$ is globally attracting if $\forall x_0 \in$ $\mathbb{R}^2$ the solution $x(t)$ of*

$$\dot{x} = f(x) \quad , \quad x(t_0) = x_0$$

satisfies

$$\lim_{t \to \infty} \| x(t) - \bar{x} \| = 0.$$

Definition 18 *A fixed point $\bar{x}$ of $\dot{x} = f(x)$ is asymptotically stable if it is both Liapunov stable and attracting.*

Definition 19 *A fixed point $\bar{x}$ of $\dot{x} = f(x)$ is neutrally stable if it is Liapunov stable but not attracting.*

Definition 20 *A fixed point $\bar{x}$ of $\dot{x} = f(x)$ is unstable if it is not stable.*

Notice that, by definition, if $\bar{x}$ is asymptotically stable it is also Liapunov stable. Also, not all attracting fixed points are necessarily asymptotically stable so that the definitions are mutually exclusive. Below we summarize these definitions in a Table:

	Attracting?	Liapunov Stable?
Asymptotic Stability	Y	Y
Neutral Stability	N	Y
Unstable	N	N
	Y	N

It should be remarked that there are different conventions in the definitions of stability throughout the literature. Some also address stability of solutions that are not fixed points. For instance, some authors define the Liapunov stability of a solution $x^*(t)$ of an initial value problem in the same way that the Liapunov stability of a fixed point is defined. In particular,

$$x_0 \in N_\delta(x^*(t_0)) \Rightarrow \| x(t) - x^*(t) \| < \epsilon \quad , \quad \forall t \geq t_0$$

This sort of issue will become important when we talk about periodic solutions of planar systems.

More importantly, depending on the source the definitions above can have different names. For example, in [7], "Liapunov Stable" is "stable" but our definition of "attracting" is equivalent to their definition of "asymptotically stable" (page 128). In our textbook [8], "stable" and "asymptotically stable" are equivalent which should be contrasted with [5] (page 266) where "stable" is equivalent to "Liapunov stable" (the convention we adopt). The author [9] has identical definitions to ours except do not have a definition for an "attracting" fixed point. In [5], "attracting" fixed points are also not defined. In [4, 6], there are also separate definitions for uniform stability and Poincare stability. Thus, when consulting other resources it is important to know which definition is being used! The definitions for "stability" are especially important since "unstable" is most often defined as "not stable". Our textbook, however, gives the definition of "unstable" as "neither attracting nor Liapunov stable" (page 129). This definition is a bit vague in my opinion. If one is to interpret that as meaning it is not Liapunov stable and it is not attracting then a fixed point that is attracting but but not Liapunov stable is not stable (by their definition) yet it is not unstable! For this and other reasons we are adopting the definitions set out in this writeup.

6 Review of Linear Algebra

Here we give an overview of some linear algebra tools and definitions needed to solve and analyze the dynamics of linear systems in the next section.

First, the imaginary number i is that number such that $i^2 = -1$ and the symbol $\mathbb{C}$ will be used to denote the space of complex numbers. Thus $z \in \mathbb{C}$ has the form

$$z = a + ib \quad , \quad a, b \in \mathbb{R}$$

We will also need the identity

$$e^z = e^a(\cos\ b + i \sin\ b)$$

which is proveable using Taylor series expansions.

For any matrix $A \in \mathbb{C}^{n \times n}$, we define the nullspace $N(A)$ of A as:

$$N(A) = \{x \in \mathbb{C}^n : Ax = 0\} \ .$$

The zero vector is always in $N(A)$. A necessary condition for $N(A)$ to be nontrivial (not only the zero vector) is that its determinant vanishes. In this case, A is not invertible.

If $A \in \mathbb{C}^{2 \times 2}$,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad , \quad a_{ij} \in \mathbb{C},$$

then its determinant is

$$det(A) = a_{11}a_{22} - a_{12}a_{21}$$

If $det(A) \neq 0$ the inverse matrix A^{-1} of $A \in \mathbb{C}^{2 \times 2}$ exists and is given by the simple formula:

$$A^{-1} = \frac{1}{det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \ .$$

For $A \in \mathbb{C}^{2 \times 2}$ ("2 by 2" matrices) the computation of the nullspace is very simple. If $det(A) = 0$ then the (two) row vectors of A are necessarily dependent so row reduction is not needed to find a spanning vector for $N(A)$. The example below illustrates the determination of such a spanning vector.

Example 1 *For the matrix*

$$A = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix} ,$$

the equation $Ax = 0$ is equivalent to

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ -6x_1 + 3x_2 &= 0 \end{aligned}$$

Setting $x_1 = 1$ one finds $x_2 = 2$ so that by defining $\vec{\zeta} = (1,2)^T$, the nullspace of A is any multiple of $\vec{\zeta}$ or

$$N(A) = span\{\vec{\zeta}\}.$$

The nullspace is the line $x_2 = 2x_1$ in $\mathbb{R}^2$.

In order to solve linear systems of differential equations we must first define eigenvalues, eigenvectors and eigenspaces. Though we will only be dealing with real matrices A we will state the definitions as if A were complex.

Definition 21 *Let $A \in \mathbb{C}^{n\times n}$. A number $\lambda \in \mathbb{C}$ is an* <u>*eigenvalue*</u> *of A if there exists an $x \in \mathbb{C}^n$, $x \neq 0$ such that $Ax = \lambda x$. Any such x associated with an eigenvalue λ is an* <u>*eigenvector*</u> *of A. Further, for any eigenvalue λ of A we define the* <u>*eigenspace*</u> *$E_\lambda(A)$ of A as:*

$$E_\lambda(A) = \{x \in \mathbb{C}^n : Ax = \lambda x\}.$$

An alternate way of thinking of eigenvalues is that they are those λ for which $N(A - \lambda I)$ is nontrivial (here I is the identity matrix). This is only possible if $det(A - \lambda I)$ vanishes. Thus, eigenvalues are roots of the <u>characteristic polynomial</u>

$$P(\lambda) \equiv det(A - \lambda I) = 0 .$$

Example 2 *Let*

$$A = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} .$$

Then

$$P(\lambda) = det\left(\begin{bmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix}\right) = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) .$$

Thus, the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = -2$. To find an eigenvector $\vec{\zeta}_1$ of A associated with the eigenvalue λ_1 note that

$$A - \lambda_1 I = \begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix}$$

so that $\vec{\zeta}_1 = (1,3)^T$ and $E_{\lambda_1}(A) = span\{(1,3)^T\}$. In $\mathbb{R}^2$ this space is the line $x_2 = 3x_1$. For the other eigenvalue λ_2 one finds $\vec{\zeta}_2 = (1,-2)^T$ and the eigenspace $E_{\lambda_2}(A) = span\{(1,-2)^T\}$ is the line $x_2 = -2x_1$.

Now we have the standard diagonalization theorem:

Theorem 11 *(Diagonalization) If $A \in \mathbb{C}^{n\times n}$ has n distinct (nonequal) eigenvalues $\lambda_1, \lambda_2, \ldots \lambda_n$ with associated eigenvectors $\vec{\zeta}_1, \vec{\zeta}_2, \ldots \vec{\zeta}_n$ then the matrix S with $\vec{\zeta}_j$ as its columns*

$$S = [\vec{\zeta}_1, \vec{\zeta}_2, \ldots \vec{\zeta}_n]$$

diagonalizes A as follows

$$S^{-1}AS = \Lambda$$

where

$$\Lambda = diag(\lambda_1, \lambda_2, \ldots \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}.$$

It can easily be verified that in our previous example the matrix

$$S = [\vec{\zeta}_1, \vec{\zeta}_2] = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}$$

diagonalizes A. Note, however, that not all matrices can be diagonalized by their eigenvectors. Indeed, the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has a characteristic polynomial

$$P(\lambda) = \lambda^2$$

and thus has only one eigenvalue $\lambda = 0$. Since $\lambda = 0$ is a double root it is said to have algebraic multiplicity 2. Noting, $E_0(A) = N(A)$ we see that the eigenspace is one dimensional and is spanned by the eigenvector $\vec{\zeta} = (1, 0)^T$. Thus, in this instance there are not two eigenvectors with which to form S!

Eigenvectors can be normalized so their Euclidean norm is 1. For our previous example,

$$\| \vec{\zeta_1} \| = \sqrt{10}$$

so that

$$\hat{\zeta}_1 = \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right)^T$$

is the normalized eigenvector associated with λ_1. Then, a symmetric [11] matrix $A \in \mathbb{R}^{n \times n}$ can be orthogonally diagonalized by an orthogonal matrix. If A is a real symmetric matrix satisfying the assumptions of the previuos Theorem then this orthogonal matrix Q is formed by the normalized eigenvectors:

$$Q = [\hat{\zeta}_1, \hat{\zeta}_2, \ldots \hat{\zeta}_n] \quad , \quad Q^{-1} A Q = \Lambda.$$

By defintion, a matrix is orthogonal if

$$Q^{-1} = Q^T$$

or that their transpose is their inverse. As a consequence

$$\| Qx \| = \| x \| \quad , \quad \forall x \in \mathbb{R}^n$$

or that they preserve length.

[11] $A^T = A$

7 Linear Planar Systems - Definition and Fixed Points

A linear system of differential equations in $\mathbb{R}^2$ is a system of the form:

$$\begin{aligned} \dot{x_1} &= a_{11}x_1 + a_{12}x_2 \\ \dot{x_2} &= a_{21}x_1 + a_{22}x_2 \end{aligned}$$

where $a_{i,j}, i, j = 1, 2$, can be functions of t. Unless otherwise stated, however, we will assume $a_{i,j}$ are constants.

Letting $x = (x_1, x_2)^T$ be a column vector with $x_1(t)$ and $x_2(t)$ as its components, the system above can be written:

$$\dot{x} = Ax \tag{67}$$

where the matrix $A \in \mathbb{R}^{2\times 2}$ is:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} .$$

Providing $det(A) \neq 0$ the system $\dot{x} = Ax$ has the sole fixed point $x = (0, 0)$. If $det(A) = 0$ then every $\bar{x} \in N(A)$ is a fixed point.

Example 3 *Let*

$$\dot{x} = Ax = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Since $det(A) = 0$ *the nullspace is nontrivial. The first row of* $Ax = 0$ *is equivalent to* $2x_1 - x_2 = 0$*. Setting* $x_1 = 1$ *yields* $x_2 = 2$ *so that* $N(A)$ *is spanned by the vector* $\zeta = (1, 2)^T$*, i.e.,*

$$N(A) = span\{(1, 2)^T\} .$$

Geometrically, $N(A)$ *is the line* $x_2 = 2x_1$*. All points on this line are fixed points of* $\dot{x} = Ax$*.*

8 Solutions to IVP of Linear Systems in the Plane

Let $p_1(t)$ and $p_2(t)$ be two two solutions of the system $\dot{x} = Ax$, where $A \in \mathbb{R}^{2\times 2}$. That is,

$$\dot{p}_i = Ap_i \quad , \quad i = 1, 2$$

where $p_i(t) \in \mathbb{R}^2$. The Wronskian of p_1 and p_2 is the determinant of the matrix whose columns are formed by the column vector solutions p_i, i.e.,

$$W(p_1, p_2) = det[p_1|p_2].$$

The two solutions p_1 and p_2 are said to be linearly independent if their Wronskian does not vanish [12] . If the solutions are linear independent then the general solution of the initial value problem

$$\dot{x} = Ax \quad , \quad x(0) = x_0$$

is given as

$$x(t) = c_1 p_1(t) + c_2 p_2(t)$$

where c_1 and c_2 are constants.

For any matrix A and vector $c = (c_1, c_2)^T$, the product Ac yields a linear combination of the columns of A. Thus, by defining the matrix

$$\Psi(t) = [p_1(t)|p_2(t)]$$

the general solution $x(t)$ can be written:

$$x(t) = \Psi(t)c\ .$$

Since the solutions forming the columns of Ψ are linearly independent, Ψ is invertible. Given the initial condition for x,

$$x(0) = x_0 = \Psi(0)c \quad \Rightarrow \quad c = \Psi(0)^{-1}x_0\ .$$

Therefore,

$$x(t) = \Phi(t)x_0 \quad , \quad \Phi(t) = \Psi(t)\Psi(0)^{-1}\ . \tag{68}$$

The matrix Φ is referred to as the Fundamental Solution Matrix for the problem although some authors also call Ψ a Fundamental Solution Matrix. There is only one Φ but the Ψ are not unique. For example one could have just as easily defined $\Psi = [ap_1|p_2]$ where a is any constant.

Since $\Phi(t)$ is unique it is sometimes written

$$\Phi(t) = e^{At}$$

where A is the original matrix defining the planar system. Precise definitions for functions of matrices ($e^A, sin(A), etc.$) is part of the subject of spectral theory (in Functional Analysis) and is usually accomplished using Taylor series. For instance, since A^n makes sense for any integer so the convergence of the series

$$e^{At} = I + tA + \frac{1}{2!}t^2A^2 + \cdots$$

[12] It can be shown [1] that the Wronskian of two solutions is either identically zero or it never vanishes

can be discussed using matrix norms. It can be shown that the series on the right does converge to the fundamental matrix $\Phi(t)$ but we do not need to discuss such issues here. Be aware, however, that some books develop the theory for linear systems using a complete development of the definition of e^{At}.

Given a matrix A, equation (68) implies the solution $x(t)$ for any initial condition x_0 can be found if one can determine $\Psi(t)$. In most instances, this this amounts to finding the eigenvalues and eigenvectors of A. To see why this is, suppose one assumes a solution of the form

$$x(t) = e^{\lambda t}\zeta \quad , \quad \zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$$

where $\lambda \in \mathbb{C}$ is some constant and $\zeta \in \mathbb{C}^2$ is some constant vector. Substituting this expression into $\dot{x} = Ax$ yields

$$\lambda e^{\lambda t}\zeta = e^{\lambda t}A\zeta$$

or

$$A\zeta = \lambda\zeta \; .$$

If λ were chosen so that the only ζ which solved this problem were $\zeta = 0$ then the resulting solution $x(t) \equiv 0$ is uninteresting. However, if λ is an eigenvalue of A then there do exist nontrivial $\zeta \in N(A - \lambda I)$. Therefore, it appears that a prerequisite for determining $\Psi(t)$ is to find all the eigenvalues and eigenvectors of A. Although this is true, some other issues complicate matters but overall the construction of the Fundamental Solution Matrix can be categorized into three classes which we discuss in the subsequent three sections.

8.1 Real, Distinct Eigenvalues

Suppose that $A \in \mathbb{R}^{2\times 2}$ has two real and distinct eigenvalues λ_1, λ_2, $\lambda_1 \neq \lambda_2$ with respective eigenvectors ζ_1 and ζ_2 (Note here that ζ_i are vectors and not components of a vector). From basic linear algebra theory it can be shown that these eigenvectors are independent and that as a result the following two solutions are linearly independent [13]

$$x_1(t) = e^{\lambda_1 t}\zeta_1 \quad , \quad x_2(t) = e^{\lambda_2 t}\zeta_2$$

[13] we omit the details

Thus, a Fundamental Solution Matrix is:

$$\Psi(t) = [e^{\lambda_1 t}\zeta_1 | e^{\lambda_2 t}\zeta_2] ,$$

from which $\Phi(t)$ can be computed.

Example 4 *Let*

$$\dot{x} = Ax = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic polynomial for A is

$$P(\lambda) = det(A - \lambda I) = (1 - \lambda)^2 - 4$$

The roots of P are the eigenvalues. In this case $\lambda_1 = 3$ and $\lambda_2 = -1$.

$$(A - \lambda_1 I) = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$$

so that $\zeta_1 = (1, 2)^T$ is an eigenvector associated with eigenvalue λ_1. Similarly, $\zeta_2 = (1, -2)^T$. The two inedepndent solutions are

$$x_1(t) = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} \quad , \quad x_2(t) = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix} .$$

A Fundamental Solution Matrix is

$$\Psi(t) = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix} ,$$

from which one finds

$$\Psi(0) = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \quad , \quad \Psi(0)^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix} \quad .$$

To find the solution of the initial value problem

$$\dot{x} = Ax \quad , \quad x(0) = x_0 = (1, 0)^T$$

note that

$$c = \Psi(0)^{-1} x_0 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

so that

$$x(t) = \Psi(t)c = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \\ e^{3t} - e^{-t} \end{pmatrix}$$

As a final note, it is possible that one of the eigenvalues is zero in this case. Suppose $\lambda_1 = 0$. Then, one solution is a constant (nontrivial) solution

$$x_1(t) = \zeta_1$$

where ζ_1 is the eigenvector associated with the zero eigenvalue. In this case, the eigenspace $E_{\lambda_1}(A) = E_0(A) = N(A)$! In other words, A is not invertible since $det(A - 0I) = 0$. These constant solutions correspond to the fixed points of $\dot{x} = Ax$ which occur on the line spanned by ζ_1.

8.2 Complex Conjugate Eigenvalues

Suppose that $A \in \mathbb{R}^{2\times 2}$ has complex eigenvalues. Such eigenvalues (being roots of a quadratic) must occur in complex conjugate pairs. Specifically, suppose that one eigenvalue λ is

$$\lambda = a + ib \quad .$$

Then there is a complex eigenvector $\zeta \in \mathbb{C}^2$ such that

$$A\zeta = \lambda\zeta$$

The complex conjugate of any complex number $z = a + ib$ is defined as:

$$\bar{z} = a - ib$$

It can easily be verified that for any two complex numbers z_1 and z_2,

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2 \quad .$$

For a vector $\zeta = (\zeta_1, \zeta_2)^T$, the conjugate $\bar{\zeta}$ is defined as:

$$\bar{\zeta} = \begin{pmatrix} \bar{\zeta}_1 \\ \bar{\zeta}_2 \end{pmatrix} \quad .$$

A similar defintion holds for matrices A but in our case, A is real so that $\bar{A} = A$.

As a result, if (λ, ζ) is an eigenvalue-eigenvector pair for A, then the calculations

$$\begin{aligned} \overline{A\zeta} &= \overline{\lambda\zeta} \\ \bar{A}\bar{\zeta} &= \bar{\lambda}\bar{\zeta} \\ A\bar{\zeta} &= \bar{\lambda}\bar{\zeta} \end{aligned}$$

show that $(\bar{\lambda}, \bar{\zeta})$ is also an eigenvalue-eigenvector pair for A.

Even though the eigenvalues and eigenvectors are complex,

$$x(t) = e^{\lambda t}\zeta$$

is still a solution of $\dot{x} = Ax$. The solution, however, is not real. To construct a Fundamental Solution Matrix, we need two linearly independent real solutions. Toward this end, we define the notations $Re(X)$ and $Im(X)$ to be the real and imaginary parts of X, respectively. Then, $x(t) = x_r(t) + ix_i(t)$ where $x_r = Re(x)$ and $x_i = Im(x)$. Substituting this into the differential equation one finds:

$$\dot{x_r} + i\dot{x_i} = Ax_r + iAx_i.$$

Since the real and imaginary parts of each side of this equation must match we see that real solutions can be extracted from the real and imaginary parts of $x(t)$. By writing the complex eigenvector ζ associated with $\lambda = a + ib$ as

$$\zeta = \zeta_r + i\zeta_i$$

where $\zeta_r = Re(\zeta)$ and $\zeta_i = Im(\zeta_i)$,

$$\begin{aligned} x(t) &= e^{(a+ib)t}(\zeta_r + i\zeta_i) \\ x(t) &= e^{at}(cos(bt) + isin(bt))(\zeta_r + i\zeta_i) \\ x(t) &= x_r(t) + ix_i(t) \end{aligned}$$

where

$$x_r(t) = e^{at}\left(cos(bt)\vec{\zeta_r} - sin(bt)\vec{\zeta_i}\right) \quad (69)$$

$$x_i(t) = e^{at}\left(sin(bt)\vec{\zeta_r} + cos(bt)\vec{\zeta_i}\right). \quad (70)$$

Then, a Fundamental Solution Matrix can be formed by using $x_r(t)$ and $x_i(t)$ as its columns:

$$\Psi(t) = [x_r(t) \mid x_i(t)]$$

Notice that if $Re(\lambda) = a = 0$, solutions remain bounded but $x = 0$ is not attracting (neutral stability). If $Re(\lambda) < 0$,

$$x(t) = c_1x_r(t) + c_2x_i(t) \to 0 \quad , \quad as\ t \to \infty$$

demonstrating $x = 0$ is attracting (and asymptotically stable). If $Re(\lambda) > 0$, then the fixed point $\bar{x} = 0$ is unstable since the solution grows without bound.

Example 5 *Let*

$$\dot{x} = Ax = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic polynomial for A *is*

$$P(\lambda) = det(A - \lambda I) = \lambda^2 + 2\lambda + 2$$

The roots of P *are the eigenvalues. In this case* $\lambda = -1 + i$ *is one eigenvalue (the other is* $\bar{\lambda} = -1 - i$ *which we don't need).*

$$(A - \lambda I) = \begin{bmatrix} 2 - i & -1 \\ 5 & -2 - i \end{bmatrix}$$

so that $\zeta = (1, 2 - i)^T$ *is a complex eigenvector associated with the eigenvalue* λ*. Here,*

$$\zeta = \zeta_r + i\zeta_i = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Using (69)-(70), one finds two independent (real) solutions:

$$x_r(t) = \begin{pmatrix} e^{-t} cos\ t \\ e^{-t}(2cos\ t + sin\ t) \end{pmatrix} \quad , \quad x_i(t) = \begin{pmatrix} e^{-t} sin\ t \\ e^{-t}(2sin\ t - cos\ t) \end{pmatrix}$$

A Fundamental Solution Matrix is

$$\Psi(t) = \begin{bmatrix} e^{-t} cos\ t & e^{-t} sin\ t \\ e^{-t}(2cos\ t + sin\ t) & e^{-t}(2sin\ t - cos\ t) \end{bmatrix},$$

from which $\Phi(t)$ *can be computed.*

8.3 Real and Equal Eigenvalues

The last case to consider is when $A \in \mathbb{R}^{2 \times 2}$ has a single repeated eigenvalue. An simple example of such a matrix is:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

whose characteristic polynomial is $P = (1 - \lambda)^2$, i.e., $\lambda = 1$ is the sole (repeated) eigenvalue.

If λ_0 is a repeated eigenvalue of $A \in \mathbb{R}^{2 \times 2}$ and ζ_0 is the associated eigenvector then

$$x(t) = e^{\lambda_0 t} \zeta_0$$

is still a solution. The problem is that we do not have another eigenvalue-eigenvector pair from which to construct a second solution [14]. To find a second solution, assume that

$$y(t) = te^{\lambda_0 t}\eta^* + e^{\lambda_0 t}\eta \tag{71}$$

where η^* and η are vectors to be determined. Straight forward calculations reveal

$$Ay - \dot{y} = te^{\lambda_0 t}\left(A\eta^* - \lambda_0\eta^*\right) + e^{\lambda_0 t}\left(A\eta - \lambda_0\eta - \eta^*\right) \ .$$

Thus, if we choose η^* and η so that

$$(A - \lambda_0 I)\eta^* = 0 \tag{72}$$

$$(A - \lambda_0 I)\eta = \eta^* \tag{73}$$

then $y(t)$ solves $\dot{y} = Ay$. Since λ_0 is an eigenvalue of A then (72) will be satisfied by the choice $\eta^* = \zeta_0$, the eigenvector. In summary, the second solution $y(t)$ is

$$y(t) = te^{\lambda_0 t}\zeta_0 + e^{\lambda_0 t}\eta \tag{74}$$

where η is a solution of

$$(A - \lambda_0 I)\eta = \zeta_0 \ . \tag{75}$$

Then the Fundamental Solution Matrix is formed in the usual way:

$$\Psi(t) = [\ x(t)\ |y(t)\ ] = [\ e^{\lambda_0 t}\zeta_0 \mid te^{\lambda_0 t}\zeta_0 + e^{\lambda_0 t}\eta\ ].$$

One key issue constructing $\Psi(t)$ in such a way is the solvability of (75). In particular, one cannot simply write $\eta = (A - \lambda_0 I)^{-1}\zeta_0$ since λ_0 was chosen so that the inverse of $(A - \lambda_0 I)$ did not exist! Nevertheless, solution η of (75) may still exist [15]. Below we illustrate the procedure in an example.

Example 6 *Let*

$$\dot{x} = Ax = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic polynomial for A is

$$P(\lambda) = det(A - \lambda I) = (\lambda - 2)^2$$

[14] except in the exceptional case where A is the zero matrix. Then, $\lambda_0 = 0$ and $(1,0)^T, (0,1)^T$ are two independent eigenvectors.

[15] They won't be unique since one can can always add an element of $N(A - \lambda_0 I)$ to η and that will still be a solution.

Thus $\lambda = \lambda_0 = 2$ is a repeated eigenvalue. Since

$$(A - 2I) = \left[\begin{array}{rr} -1 & -1 \\ 1 & 1 \end{array} \right] ,$$

$\zeta_0 = (1, -1)^T$ is an eigenvector associated with the eigenvalue λ_0. Thus,

$$x(t) = e^{2t}\zeta_0 = \left(\begin{array}{c} e^{2t} \\ -e^{2t} \end{array} \right)$$

is a solution. To find $y(t)$ in (74) we need to find a solution η of $(A - 2I)\eta = \zeta_0$. If $\eta = (\eta_1, \eta_2)^T$, this is the same as finding a solution of :

$$\left[\begin{array}{rr} -1 & -1 \\ 1 & 1 \end{array} \right] \left(\begin{array}{c} \eta_1 \\ \eta_2 \end{array} \right) = \left(\begin{array}{r} 1 \\ -1 \end{array} \right)$$

As an augmented matrix this system is:

$$[A - 2I|\zeta_0] = \left[\begin{array}{rrr} -1 & -1 & 1 \\ 1 & 1 & -1 \end{array} \right]$$

which after row reduction yields:

$$\left[\begin{array}{rrr} -1 & -1 & 1 \\ 0 & 0 & 0 \end{array} \right]$$

As a scalar equation this is equivalent to:

$$-\eta_1 - \eta_2 = 1$$

so that if $\eta_1 = -1$, we must have $\eta_2 = 0$ or

$$\eta = (-1, 0)^T .$$

Then, $y(t)$ is known:

$$y(t) = \left(\begin{array}{c} (t-1)e^{2t} \\ -te^{2t} \end{array} \right)$$

Then, a Fundamental Matrix Solution is

$$\Psi(t) = [\ x(t)\ |y(t)\ ] = \left[\begin{array}{cc} e^{2t} & (t-1)e^{2t} \\ -e^{2t} & -te^{2t} \end{array} \right]$$

Notice how the growth of $y(t)$ is faster than the growth of $x(t)$ since the exponential is multiplied by t.

8.4 Basic Linear Subspaces for Fixed Points

For the planar system

$$\dot{x} = Ax \quad , \quad A \in \mathbb{R}^{2\times 2}$$

the solution $x(t)$ and fixed point stability properties can all be determined from the eigenvalues and eigenvectors of A. If

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

the characteristic polynomial

$$P(\lambda) = det(A - \lambda I) = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$$

Written another way,

$$P(\lambda) = \lambda^2 - Tr(A)\lambda + det(A)$$

where $Tr(A) = a_{11} + a_{22}$ is the trace of the matrix A. Thus, the stability of the fixed point $\bar{x}$ is determined entirely by the two quantities $Tr(A)$ and $det(A)$. Roots of $P(\lambda)$ are:

$$\lambda_{\pm} = \frac{1}{2}\left(Tr(A) \pm \sqrt{(Tr(A))^2 - 4det(A)}\right)$$

By considering all the possible permutations of signs of $Tr(A)$ and $(Tr(A))^2 - det(A)$ one can easily deduce the following table for the stability of $\bar{x} = 0$.

Associated with $\bar{x} = 0$ we also define three linear manifolds:

Definition 22 *For $A \in \mathbb{R}^{2\times 2}$, let*

$$A\xi_k = \lambda_k \xi_k \quad , \quad \xi_k = x_k + iy_k \quad , \quad k = 1, 2$$

where x_k and y_k are the real and imaginary parts of the eigenvectors ξ_k when they are complex. Then,

$$\begin{aligned} E^s(0) &\equiv span\{x_k, y_k : Re(\lambda_k) < 0\} \\ E^c(0) &\equiv span\{x_k, y_k : Re(\lambda_k) = 0\} \\ E^u(0) &\equiv span\{x_k, y_k : Re(\lambda_k) > 0\} \end{aligned}$$

Here $E^s(0), E^c(0)$ and $E^u(0)$ are the linear stable, center and unstable manifolds associated with $\bar{x} = 0$.

$det(A) < 0$	$\bar{x} = 0$ is a saddle
$0 < det(A) \leq \frac{1}{4}(TrA)^2, TrA < 0$	$\bar{x} = 0$ is a stable node
$det(A) > \frac{1}{4}(TrA)^2, TrA < 0$	$\bar{x} = 0$ is a stable spiral
$det(A) > 0, TrA = 0$	$\bar{x} = 0$ is a center
$det(A) > \frac{1}{4}(TrA)^2, TrA > 0$	$\bar{x} = 0$ is an unstable spiral
$0 < det(A) \leq \frac{1}{4}(TrA)^2, TrA > 0$	$\bar{x} = 0$ is an unstable node
$det(A) = 0$	$\bar{x} \in N(A)$ are all fixed points

9 Planar Systems: Linearization

In this section we define the linearization of the planar system

$$\dot{x_1} = f_1(x_1, x_2) \tag{76}$$
$$\dot{x_2} = f_2(x_1, x_2) \tag{77}$$

about a fixed point. Recall this system can be written

$$\dot{x} = f(x)$$

where

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \quad , \quad f(x) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} \quad .$$

For any smooth $x(t)$, $\dot{x}(t)$ is a vector tangent to the curve parameterized by $x(t) = (x_1(t), x_2(t))$. Thus, if $x(t)$ is any solution of $\dot{x} = f(x(t))$, it must be tangent to the vector field defined by f. This means that the vector field alone $f(x)$ determines the "flow" of solutions in the plane.

If f is sufficiently smooth (i.e., if each component is continuously differentiable) then solutions exist for a finite time and are unique. One immediate consequence is that trajectories cannot cross in the (x_1, x_2) phase plane. The reason for this is that such a crossing would contradict the uniqueness of solutions. For example, if two different trajectories crossed at $x_0 \in \mathbb{R}^2$ then there would be two different solutions to $\dot{x} = f(x), x(0) = x_0$!

Also note that if "blowup" occurs it cannot be observed in a phase portrait. Blowup means that $\| x(t) \| \to \infty$ as $t \to T^-$ where T is some finite time. If there were blowup it could not be drawn in the (x_1, x_2) plane since either or both of x_1, x_2 is approaching infinity. Also, just from looking at a trajectory drawn in the (x_1, x_2) plane there is no way to determine how fast $x(t)$ is moving. This could be ascertained from the vector field f since $\| f(x) \|$ is the speed of the solution $x(t)$. Even so, that information alone would not be enough to determine if the approach of a trajectory to ∞ occurs in <u>finite</u> time.

Definition 23 *For the system (76)-(77), we define the x_1-nullcline (or $\dot{x}_1$-nullcline) as the set of (x_1, x_2) such that $f_1(x_1, x_2) = 0$. A similar definition holds for x_2-nullcline. The set (x_1, x_2) such that $f_1(x_1, x_2) = k$ will be referred to as the $\dot{x}_1 = k$ isocline of x_1.*

Note that on the x_1-nullcline,

$$f = \begin{pmatrix} 0 \\ f_2 \end{pmatrix} \quad ,$$

so that trajectories flow upward or downward. On x_2-nullclines, the "flow" is horizontal.

Fixed points of $\dot{x} = f(x)$ occur where nullclines intersect since a fixed point $\bar{x} = (\bar{x}_1, \bar{x}_2)$ satifies the coupled equations

$$\begin{aligned} f_1(\bar{x}_1, \bar{x}_2) &= 0 \\ f_2(\bar{x}_1, \bar{x}_2) &= 0 \end{aligned}$$

The word "flow" has a well defined meaning in the context of ordinary differential equations.

Definition 24 *The flow $\phi(t, x_0)$ generated by the differential equation $\dot{x} = f(x)$, $(x(t) \in \mathbb{R}^n)$ is that function $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ which solves the initial value problem*

$$\dot{x} = f(x) \quad , \quad x(0) = x_0.$$

Specifically,

$$\frac{\partial \phi}{\partial t} = f(\phi(t, x_0))$$

and

$$\phi(0, x_0) = x_0.$$

Example 7 *For the linear planar system $\dot{x} = Ax$ the flow ϕ can be found explicitly using the Fundamental Solution Matrix $\Phi(t)$ as follows:*

$$\phi(t, x_0) = \Phi(t) x_0.$$

In particular, this defines a map from $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}^n$.

The flow ϕ generated by a differential equation has two other noteable properties. These are [16]

a) $\phi(t + s, x_0) = \phi(t, \phi(s, x_0))$

b) For each fixed t the map $\phi(t, \cdot)$ is invertible with inverse $\phi(-t, \cdot)$. That is, $\phi(-t, \phi(t, x_0)) = x_0$.

[16] whenever the expressions are defined

9.1 Higher Order Taylor Series: Jacobians, Hessians

If a function $f : \mathbb{R}^2 \to \mathbb{R}$ is sufficiently smooth near some point $(\bar{x}, \bar{y})$ then it has an m-th order Taylor series expansion which converges to the function as $m \to \infty$. Expressions for m-th order expansions are complicated to write down. For our purposes we will only need second order expansions so we state a related Theorem here:

Theorem 12 *Let $f : \mathbb{R} \to \mathbb{R}$ and assume that f and all its derivatives up to second-order are continous on some neighbourhood $N_r(a)$, $a = (a_1, a_2)$. For each $x = (x_1, x_2) \in N_r(a)$ there exists a ζ (which depends on x) on the line connecting a and x such that*

$$\begin{aligned} f(x) &= f(a) + f_{x_1}(a)(x_1 - a_1) + f_{x_2}(a)(x_2 - a_2) \\ &+ \frac{1}{2!}\left(f_{x_1x_1}(a)(\zeta_1 - a_1)^2 + 2f_{x_1x_2}(a)(\zeta_1 - a_1)(\zeta_2 - a_2) + f_{x_2x_2}(a)(\zeta_2 - a_2)^2\right) \end{aligned}$$

Notationally there are many ways to write out Taylor series. For $f = f(x_1, x_2)$ one can define the gradient of f as

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right)$$

The Hessian $H_f(x)$ of f is defined as

$$H_f(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 x_2} \\ \frac{\partial^2 f}{\partial x_2 x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

Note that H_f is a symmetric matrix.

With these definitions, the expansion in the Theorem above can be written

$$f(x) = f(a) + \nabla f(a) \cdot (x - a) + \frac{1}{2!}(\zeta - a)^T H_f(a)(\zeta - a)$$

where $\cdot$ is the dot product.

Here, the linear approximation [17]of f is

$$f(x) \simeq f(a) + \nabla f(a) \cdot (x - a)$$

and the remainder term is $R_2(\zeta) = \frac{1}{2!}(\zeta - a)^T H_f(a)(\zeta - a)$. The second-order Taylor series of f about a is

$$f(x) = f(a) + \nabla f(a) \cdot (x - a) + \frac{1}{2!}(x - a)^T H_f(a)(x - a) + R_3$$

[17] or first-order Taylor series approximation

where the exact form of the remainder term R_3 is a complicated expression. Here we will simply write:

$$R_3 = O(\| x - a \|^3) \ .$$

Example 8 *Let* $f(x) = x_1^2 + e^{x_1} - x_1 x_2 + 3x_2$ *and* $a = (0, 0)$. *Then,*

$$\nabla f(x) = (2x_1 + e^{x_1} - x_2, -x_1 + 3)$$

so that

$$\nabla f(x) = (1, 3)$$

The Hessian is

$$H_f(x) = \begin{bmatrix} 2 + e^{x_1} & -1 \\ -1 & 0 \end{bmatrix}$$

so that

$$H_f(a) = \begin{bmatrix} 3 & -1 \\ -1 & 0 \end{bmatrix}$$

Since $f(a) = 1$,

$$f(x) = 1 + (1, 3) \cdot x + \frac{1}{2!} x^T \begin{bmatrix} 3 & -1 \\ -1 & 0 \end{bmatrix} x + O(\| x \|^3)$$

Longhand,

$$f(x) = 1 + x_1 + 3x_2 + \frac{3}{2} x_1^2 - x_1 x_2 + O(\| x \|^3)$$

Expansions for a vector valued function $f : \mathbb{R}^2 \to \mathbb{R}^2$ are obtained by expanding each component separately. Thus, if $f = (f_1, f_2)$,

$$f(x) = \begin{pmatrix} f_1(a) \\ f_2(a) \end{pmatrix} + \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a)(x_1 - a_1) + \frac{\partial f_1}{\partial x_2}(a)(x_2 - a_2) \\ \frac{\partial f_2}{\partial x_1}(a)(x_1 - a_1) + \frac{\partial f_2}{\partial x_2}(a)(x_2 - a_2) \end{pmatrix} + \cdots$$

By defining the <u>Jacobian</u> $Df(x)$ of f at x as the matrix:

$$Df(x) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) \end{bmatrix}$$

then the Taylor series above can be written

$$f(x) = f(a) + Df(a)(x - a) + \cdots$$

The second order expansion of f can be written

$$f(x) = f(a) + Df(a)(x-a) + \frac{1}{2!}(x-a)^T H_f(a)(x-a) + O(\| x \|^3)$$

where the term involving the Hessian must be interpretted [18]

$$(x-a)^T H_f(a)(x-a) = \begin{pmatrix} (x-a)^T H_{f_1}(a)(x-a) \\ (x-a)^T H_{f_2}(a)(x-a) \end{pmatrix}$$

9.2 Linearization Process

Suppose that

$$\dot{x} = f(x)$$

has a fixed point $\bar{x} = (x_1, x_2)$. Near $\bar{x}$, we expect that a reasonable approximation of $x(t)$ can be obtained using a first order Taylor series of f about $\bar{x}$. Toward this end, let

$$x(t) = \bar{x} + \eta u(t)$$

where the parameter $\eta \ll 1$ has been introduced to emphasize that the difference $\| x(t) - \bar{x} \|$ is small, i.e., $\| x(t) - \bar{x} \| = O(\eta)$. Since $\bar{x}$ is constant, $\dot{x} = \eta \dot{u}$ so that

$$\eta \dot{u} = f(\bar{x} + \eta u)\ .$$

So far this equation is an exact expression for $u(t)$. Now we expand the right side in a Taylor series about $\bar{x}$ as follows:

$$\begin{aligned} \eta \dot{u} &= f(\bar{x}) + \eta Df(\bar{x})u + \frac{1}{2!}\eta^2 u^T H_f(\bar{x})u + \cdots \\ &= \eta Df(\bar{x})u + \frac{1}{2!}\eta^2 u^T H_f(\bar{x})u + \cdots \end{aligned}$$

where the last step is due to the fact that $f(\bar{x}) = 0$. Cancelling η yields:

$$\dot{u} = Df(\bar{x})u + \frac{1}{2!}\eta u^T H_f(\bar{x})u + \cdots$$

Were the exact remainder term included, this equation for $u(t)$ would also be exact. Now we define the linearization of $\dot{x} = f(x)$ about the fixed point to be the system:

$$\dot{v} = Df(\bar{x})v \tag{78}$$

[18] When f is vector valued, H_f is a tensor - not a matrix

This equation is also referred to as the linear variational equation of $\dot{x} = f(x)$ about $\bar{x}$. We have deliberately used $v(t)$ instead of $u(t)$ to emphasize that u and v may not be exactly equal.

A natural question to ask is to what extent does the linearization (78) predict the qualitative and/or quantitative behavior of the true variation $u(t)$ (and hence $x(t)$)? There are many theorems which address this question and give very precise answers. For our purposes we need only know that the approximation is very good for most fixed points. Which ones?

Definition 25 *A fixed point $\bar{x}$ of $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ is hyperbolic if all the eigenvalues of $Df(\bar{x})$ have nonzero real parts. If $\bar{x}$ is not hyperbolic, it is said to be nonhyperbolic.*

Note that centers are nonhyperbolic but that nodes, spirals and saddles are all hyperbolic. Also, linear systems $\dot{x} = Ax$ where $N(A)$ is nontrivial have a line of nonhyperbolic fixed points.

Now we can state that if $\bar{x}$ is hyperbolic the linearization (78) well approximates the quantitative and qualitative behavior. Again, there are many separate theorems which address these issues in a precise manner [19]. In particular, for every $\epsilon < 1$ there exists a $\delta > 0$ and a $T > 0$ such that $x_0 \in N_\delta(\bar{x})$ implies

$$\| \phi(t, x_0) - v(t) \| < \epsilon \quad , \quad \forall t \in (-T, T)$$

where $v(t)$ is the solution of the initial value problem

$$\dot{v} = Df(\bar{x}) \quad , \quad v(0) = x_0 - \bar{x} \ .$$

This is more a statement of the quantitative closeness of solutions. Essentially if your initial condition x_0 is close to a hyperbolic fixed point $\bar{x}$ then the true solution $x(t)$ will be close to $\bar{x} + v(t)$ - at least for $t \in (-T, T)$.

Earlier we defined the concept of "topological equivalence" for dynamics on $\mathbb{R}$. An analagous definition holds for dynamics on $\mathbb{R}^n$. Given such a definition, the goal of theorems relating qualitatively similarities between the true solution $x(t)$ and the variation $v(t)$ determined by the linearized problem revolve around showing a "near" topological equivalence. Essentially, what one does there is to show that the vector fields $f(x)$ and $Df(\bar{x})v$ are nearly the same everywhere near the fixed point. Here we do not delve into the subtle issues surrounding such theorems.

[19] a noteable one being the stable manifold theorem

Instead, we will use our fixed point classifcation scheme for linear systems to classify critical points of the nonlinear problems. Here we summarize the classification of fixed point of $\dot{x} = Ax$ making the identification $A = Df(\bar{x})$ and acknowledge the possibility that when $\bar{x}$ is nonhyperbolic the behavior near $\bar{x}$ may not be determined by the linearization and that in such an event, higher order terms in the Taylor series approximation may become relevant.

$det(A) < 0$	$\bar{x}$ is locally a saddle
$0 < det(A) \leq \frac{1}{4}(TrA)^2, TrA < 0$	$\bar{x}$ is locally a stable node
$det(A) > \frac{1}{4}(TrA)^2, TrA < 0$	$\bar{x}$ is locally a stable spiral
$det(A) > 0, TrA = 0$	$\bar{x}$ is locally a center
$det(A) > \frac{1}{4}(TrA)^2, TrA > 0$	$\bar{x}$ is locally an unstable spiral
$0 < det(A) \leq \frac{1}{4}(TrA)^2, TrA > 0$	$\bar{x}$ is locally an unstable node
$det(A) = 0$	$\bar{x}$ is degenerate

10 Topological Equivalence and failures

Here we make some definitions to more accurately define how the flow of nonlinear systems near fixed points are "equivalent" to their linearizations and give an example of failure of equivalence.

Definition 26 *A homeomorphism is a continuous map $H : X \to Y$ with continuous inverse. In this case we say that H is a homeomorphism from X into Y. When $X = \mathbb{R}^n$ and $Y = \mathbb{R}^n$, a homeomorphism H which is continuously differentiable is a diffeomorphism.*

Given this definition we can now precisely define "equivalent" flows. Roughly speaking, if there is an invertible transformation which maps the flow from one system onto the other then the flows are "topologically equivalent". The exact definition below for topological equivalence can easily be generalized to flows on $\mathbb{R}^n$.

Definition 27 *Let the flows of the systems*

$$\dot{x} = f(x) \quad , \quad x(0) = x_0 \quad , \quad x(t) \in \mathbb{R}^2 \tag{79}$$

$$\dot{y} = g(y) \quad , \quad y(0) = y_0 \quad , \quad y(t) \in \mathbb{R}^2 \tag{80}$$

be $\phi(t, x_0)$ and $\psi(t, x_0)$, respectively. Let $\phi(t, x_0)$ be defined for $x_0 \in U \subset \mathbb{R}^2$. We say that the systems (79)-(80) (or flows generated by) are topologically equivalent on U if there exists a homeomorphism H from U into $V = H(U)$ and a (time) interval I such that

$$H(\phi(t, x_0)) = \psi(t, H(x_0)) \quad , \quad \forall x_0 \in U \quad , \forall t \in I$$

and that flow direction is preserved.

The definition states that when systems are topologically equivalent trajectories in x-phase space can be used to compute trajectories in y-phase space via H (and vice versa). Moreover, all of the qualitative aspects of such trajectories must be retained. The statement "flow direction is preserved" imples that if $\bar{x}$ is an attracting fixed point in x phase space then $\bar{y} = H(\bar{x})$ must be attracting in y phase space. The interval I is introduced since flows may be topologically equivalent only for a finite time interval. Outside this interval the flows may be qualitatively very different. The set U is introduced since flows may be topologically equivalent only in a neighbourhood U (or local to) of some fixed point $\bar{x}$.

Example 9 *Let $\dot{x} = Ax$ where $x(t) \in \mathbb{R}^2$ and define the transformation*

$$y = Hx \quad , \quad H = \begin{bmatrix} cos\ \alpha & -sin\ \alpha \\ sin\ \alpha & cos\ \alpha \end{bmatrix}$$

H is a homeomorphism [20] from $\mathbb{R}^2$ into $\mathbb{R}^2$ which represents a counter-clockwise rotation (by angle α) of the flow $\phi(t, x_0)$ generated by $\dot{x} = Ax$

In particular $det(H) = 1$ implies the transformation is invertible. Since

$$\dot{y} = H\dot{x}$$

the associated differential equation for $y(t)$ is

$$\dot{y} = HAH^{-1}y$$

and the flow $\psi(t, y_0)$ generated is topologically equivalent to ϕ.

To see why H represents a rotation, let

$$x = (x_1, x_2)^T = (r\ cos\ \theta, r\ sin\ \theta)^T$$

where (r, θ) is polar coordinates. Here, both r and θ are functions of time t.

Given the definition of H,

$$y = Hx = \begin{bmatrix} cos\ \alpha & -sin\ \alpha \\ sin\ \alpha & cos\ \alpha \end{bmatrix} \begin{pmatrix} r\ cos\theta \\ r\ sin\theta \end{pmatrix} = \begin{pmatrix} r\ cos(\theta + \alpha) \\ r\ sin(\theta + \alpha) \end{pmatrix}.$$

The last step is accomplished by using trigonometric addition angle formulae after multiplying out the expression Hx. Notice that $y = (y_1, y_2)^T$ above is the vector x rotated a angle α counterclockwise.

Example 10 *A (nonlinear) transformation $y = H(x)$ can be defined by*

$$\begin{aligned} y_1 &= H_1(x_1, x_2) = x_2 - e^{x_1} \\ y_2 &= H_2(x_1, x_2) = x_2 \end{aligned}$$

To see why this transformation is invertible, notice that the first equation can be written

$$x_2 = y_1 + e^{x_1}$$

[20] We haven't proven H is continuous which is something we will not worry about in this course. We will focus solely on the invertibility of transformations. It is a fact, however, that if a transformation H is linear defined by the matrix multiplication $y = H(x) = Ax$, then H is continuous. Such transformations are invertible if the inverse of A exists, i.e., $det(A) \neq 0$.

For fixed $y = (y_1, y_2)$ this curve in the (x_1, x_2) plane intersects the $x_2 = y_2$ line at exactly <u>one</u> point. To find the inverse $x = H^{-1}(y)$, notice that $H_2 - H_1 = e^{x_1}$ so that the inverse is defined by:

$$
\begin{aligned}
x_1 &= log(y_2 - y_1) \\
x_2 &= y_2
\end{aligned}
$$

Notice that the transformation is only defined for y with $y_2 - y_1 > 0$. In effect, H maps $\mathbb{R}^2$ into the portion of $\mathbb{R}^2$ with $y_2 - y_1 > 0$.

The homeomorphism H can be used to create a topologically equivalent system. For example, if

$$\dot{x} = f(x)$$

then the chain rule for $y = H(x)$ is

$$\dot{y} = DH(x)\dot{x}$$

where $DH(x)$ is the Jacobian of H, i.e.,

$$DH(x) = \begin{bmatrix} \frac{\partial H_1}{\partial x_1} & \frac{\partial H_1}{\partial x_2} \\ \frac{\partial H_2}{\partial x_1} & \frac{\partial H_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} -e^{x_1} & 1 \\ 0 & 1 \end{bmatrix}$$

Given $\dot{x} = f(x)$,

$$\dot{y} = DH(x)f(x)$$

and since $x = H^{-1}(y)$,

$$\dot{y} = g(y) \equiv DH(H^{-1}(y))f(H^{-1}(y)) \tag{81}$$

For the specific transformation H,

$$DH(H^{-1}(y)) = \begin{bmatrix} -e^{log(y_2 - y_1)} & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} y_1 - y_2 & 1 \\ 0 & 1 \end{bmatrix}$$

To illustrate this procedure consider the system

$$\dot{x} = f(x) = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \tag{82}$$

where $x = (x_1, x_2)^T$. Using the inverse transformation $H^{-1}(y)$ in (81) one finds

$$\dot{y} = \begin{bmatrix} y_1 - y_2 & 1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} y_2 \\ -log(y_2 - y_1) \end{pmatrix}$$

Figure 4: Figure shows some trajectories for (83)

Expanding this out

$$\dot{y} = g(y) = \begin{pmatrix} g_1(y_1, y_2) \\ g_2(y_1, y_2) \end{pmatrix} = \begin{pmatrix} y_2(y_1 - y_2) - log(y_2 - y_1) \\ -log(y_2 - y_1) \end{pmatrix} \quad (83)$$

Since H is a homeomorphism, the systems (82) and (83) are topologically equivalent.

It is easy to verify that $\bar{x} = (0, 0)$ is a center for (82) and that all trajectories lie on circles, i.e.,

$$\frac{d}{dt}\left(x_1^2 + x_2^2\right) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 0$$

so that $x_1(t)^2 + x_2(t)^2$ is constant.

The fixed point $\bar{x}$ in the x phase plane is mapped to $\bar{y} = H(\bar{x}) = (-1, 0)^T$, It can be verified that $\bar{y}$ is also a center for (83) but trajectories are no longer closed circles. They are, however, closed orbits and periodic like those of (82). This fact would not have been immediately obvious from (83) alone. Two (closed) trajectories of (83) are shown above with the y-nullcines superimposed.

A precise statement relating the equivalence of the nonlinear flow of $\dot{x} = f(x)$ near a **hyperbolic** fixed point $\bar{x}$ is given in the theorem below.

Theorem 13 Hartman-Grobman Theorem: *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function on some open set $E \subset \mathbb{R}^n$. Let $\phi(t, x_0)$ be the flow for*

$$\dot{x} = f(x) \quad , \quad x(0) = x_0 \in E \quad (84)$$

and suppose that $\bar{x} \in E$ *is a* <u>*hyperbolic*</u> *fixed point of (84). The linearization of (84) about* $\bar{x}$ *is*

$$\dot{y} = Df(\bar{x})y \quad , \quad y(0) = y_0$$

with associated flow $\psi(t, y_0) = \Phi(t)y_0$*, where* $\Phi(t)$ *is the Fundamental Solution Matrix with* $\Phi(0) = I$*, the identity.*

Then, there exists a homeomorphism H *of an open set (neighbourhood)* U *containing* $\bar{x}$ *onto an open set* V *containing* $\bar{y} = H(\bar{x})$ *and a (time) interval* I *such that*

$$H(\phi(t, x_0)) = \psi(t, H(x_0)) \quad , \quad \forall x_0 \in U \quad , \quad \forall t \in I.$$

Moreover, H *can be chosen so that parametrization is preserved.*

The proof of this Theorem is long and yucky (technical term). The point is that near a <u>hyperbolic</u> fixed point the nonlinear (planar) system is approximated accurately by the associated linear system

$$\dot{y} = Df(\bar{x})y \quad , \quad y \in \mathbb{R}^2$$

If the linear system has a saddle, node or spiral (stable or unstable) then the flow of the nonlinear system is qualitatively the same near $\bar{x}$. Of course, further away from the fixed point solutions $\dot{x} = f(x)$ may behave very differently. The theorem <u>does NOT</u> apply if $\bar{x}$ is a center or $det(Df(\bar{x})) = 0$ (the degenerate case with a line of fixed points). In both of these cases a higher-order Taylor series approximation near $\bar{x}$ is required to approximate the flow (a case we will not examine). Instead, we will illustrate by way of example what can wrong when $\bar{x}$ is a center (nonhyperbolic). First, we illustrate how to convert planar systems to polar coordinates.

10.1 Conversion to Polar Coordinates

Here we illustrate how to convert a system

$$\dot{x} = f_1(x, y) \tag{85}$$

$$\dot{y} = f_2(x, y) \tag{86}$$

into one in polar coorinates:

$$x = r\cos\theta \quad , \quad y = r\sin\theta \quad .$$

Again, since (x, y) depend on time t, so do r and θ. Differentiating these expressions above we find:

$$\begin{aligned}\dot{x} &= \cos\theta\,\dot{r} - r\sin\theta\,\dot{\theta} = f_1(x,y)\\ \dot{y} &= \sin\theta\,\dot{r} + r\cos\theta\,\dot{\theta} = f_2(x,y)\end{aligned}$$

which can be written in the matrix form:

$$\begin{bmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{bmatrix}\begin{pmatrix}\dot{r}\\ \dot{\theta}\end{pmatrix} = \begin{pmatrix}f_1(r\cos\theta, r\sin\theta)\\ f_2(r\cos\theta, r\sin\theta)\end{pmatrix}$$

where the matrix

$$R = \begin{bmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{bmatrix}$$

has the inverse

$$R^{-1} = \begin{bmatrix}\cos\theta & \sin\theta\\ \frac{-\sin\theta}{r} & \frac{\cos\theta}{r}\end{bmatrix} \quad .$$

Thus,

$$\begin{pmatrix}\dot{r}\\ \dot{\theta}\end{pmatrix} = \begin{bmatrix}\cos\theta & \sin\theta\\ \frac{-\sin\theta}{r} & \frac{\cos\theta}{r}\end{bmatrix}\begin{pmatrix}f_1(r\cos\theta, r\sin\theta)\\ f_2(r\cos\theta, r\sin\theta)\end{pmatrix}$$

Multiplied out this is:

$$\dot{r} = \cos\theta\, f_1 + \sin\theta\, f_2 \tag{87}$$

$$\dot{\theta} = -\frac{\sin\theta\, f_1}{r} + \frac{\cos\theta\, f_2}{r} \tag{88}$$

10.2 Local Dynamics near Centers- Counterexample

Here we present an example which shows that some nonlinear systems whose linearized system has a center may in fact have qualitatively different flow near its fixed point. Consider the system

$$\dot{x} = f_1(x,y) = -y + xF(x^2+y^2) \tag{89}$$

$$\dot{y} = f_2(x,y) = x + yF(x^2+y^2) \tag{90}$$

where F is some scalar function. For example, if $F(z) = sin(z)$ then the system above would be

$$\begin{aligned}\dot{x} &= f_1(x,y) = -y + x\sin(x^2+y^2)\\ \dot{y} &= f_2(x,y) = x + y\sin(x^2+y^2)\end{aligned}$$

If we apply the conversion to polar-coordinates in the previous subsection the system (89)-(90) has the very simple (decoupled) form:

$$\begin{aligned}\dot{r} &= rF(r^2) \\ \dot{\theta} &= 1\end{aligned}$$

where $r^2 = x^2 + y^2$. So long as $F(0) = 0$, it is easy to show that the Jacobian of (89)-(90) near the fixed point $\bar{x} = (0, 0)^T$ is

$$Df(\bar{x}) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

so that the linearization of (89)-(90) has purely imaginary eigenvalues and a center at the origin.

We now show that for some F, solutions actually spiral away from the origin! Choosing $F(z) = z(1 - z)$ the system is

$$\begin{aligned}\dot{x} &= f_1(x, y) = -y + x(x^2 + y^2)(1 - x^2 - y^2) \\ \dot{y} &= f_2(x, y) = x + y(x^2 + y^2)(1 - x^2 - y^2)\end{aligned}$$

which in polar coordinates is

$$\begin{aligned}\dot{r} &= r^3(1 - r^2) = G(r) \\ \dot{\theta} &= 1\end{aligned}$$

Notice that $G(r) > 0$ for $r \in (0, 1)$. This means that if the initial conditions (x_0, y_0) are chosen so that the initial distance from the origin $r_0 = \sqrt{x_0^2 + y_0^2}$ is in $(0, 1)$, the trajectory $(x(t), y(t))$ will move away from the origin! This is NOT the same qualitative behavior of a center. The linearized system predicts that near the origin the system should oscillate when it fact steadily moves away from the origin until $r(t) \to 1^-$. The phase portrait of the flow with its nullclines is shown in the next figure.

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Figure 5: Figure shows a system whose linearized problem has a center but the true solution in fact spirals outward. Solutions tend to $r = 1$ which is the sole (limit cycle) periodic solution for the flow

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