Dynamical system examples and overview

Ordinary Differential Equations (ODE)

Most n^{th} order ODEs can be cast as:

$$(1) \qquad \frac{dx}{dt} = f(x,\mu)$$

for some $f(x,\mu)$ where $x(t) \in \mathbb{R}^n$

$$x = \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

and $\mu \in \mathbb{R}^m$ is a parameter.

Goal: seek equilibria, periodic orbits and other unique solns (chaotic) and their respective stability as a function of parameters μ.

Maps

For simplicity we define a one dimensional map

$$(2) \qquad x_{n+1} = f(x_n,\mu) \qquad n = 0,1,2\ldots \qquad x_n \in \mathbb{R}$$

Here for some initial condition x_0 eqn (2) defines an orbit with discreet time.

$$x = \{x_0, f(x_0), f^2(x_0), \ldots\}$$

The goal is the same.

ODE terminology, notation and examples

If $x = (x_1, \ldots x_n)$ and

$$(1) \qquad \frac{dx}{dt} = f(x, \mu)$$

then (1) is an n^{th} order autonomous system.
If f depends on t as in

$$(2) \qquad \frac{dx}{dt} = f(x, t, \mu)$$

then it is an n^{th} order nonautomous system

In either case the system is linear if there is a matrix $A(t) \in \mathbb{R}^{n \times n}$ and a vector $b(t) \in \mathbb{R}^n$:

$$(3) \qquad \frac{dx}{dt} = A(t)x + b(t)$$

The matrix A may be constant.

When $x(0)$ is specified, we say

$$\frac{dx}{dt} = f(x, \mu) \qquad x(0) = x_0$$

is an Initial Value Problem (IVP)

In what follows we show how one can convert sets of ODE's into systems like (1)

Examples and conversion to systems

EXAMPLE First order linear autonomous

$$\frac{dx}{dt} = \mu x \qquad x \in \mathbb{R}$$

EXAMPLE First order nonlinear autonomous

$$\frac{dx}{dt} = \mu x^2 + x \qquad x \in \mathbb{R}$$

EXAMPLE Second order linear

(1) $$y'' + y = \cos t \qquad (\ )' = \frac{d}{dt}(\ )$$

Convert to a system by letting

(2) $$x_1 = y \qquad x_2 = y' = x_1'$$

Noting $y'' = x_2'$ then (1)-(2) $\Rightarrow$

(3) $$\boxed{\begin{aligned} x_1' &= x_2 \\ x_2' &= -x_1 + \cos t \end{aligned}}$$ equivalent 2nd order linear system

For $x = (x_1, x_2)$, system (3) can be written

$$x' = Ax + b(t) = f(x,t)$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad b(t) = \begin{pmatrix} 0 \\ \cos t \end{pmatrix}$$

Conversion to autonomous.

Any nonautonomous system can be converted to an autonomous one

(1) $\dot{x} = f(x,t)$ $\qquad\qquad \dot{(\ )} = \frac{d}{dt}(\ )$

where $x \in \mathbb{R}^n$. Just let $y = t$ in

$$z = \begin{pmatrix} x \\ y \end{pmatrix}$$

Then

no "t"

$$\dot{z} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f(x,y) \\ 1 \end{pmatrix} = F(z)$$

EXAMPLE $\quad \ddot{y} + y^3 = \cos t$

Merely let

$$x_1 = y \qquad x_2 = \dot{y} \qquad x_3 = t$$

Differentiating these in t

$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = \ddot{y} \qquad \dot{x}_3 = 1$$

Hence

$$\left|\begin{array}{lcl} \dot{x}_1 & = & x_2 \\ \dot{x}_2 & = & -x_1^3 + \cos(x_3) \\ \dot{x}_3 & = & 1 \end{array}\right| \Bigg\} \text{ defines } F(z) \text{ above.}$$

Can't convert to systems always

(1) $(\ddot{y})^3 - \dot{y}\ddot{y} + y = 0$

If we let $x_1 = y$ and $x_2 = \dot{y}$ then (1) becomes

(2) $(\dot{x}_2)^3 - x_2\dot{x}_2 + x_1 = 0$

Here we can't solve for $\dot{x}_2$ (easily) but we have the more general statement

$$F(x, \dot{x}) = 0 \qquad x \in \mathbb{R}^2$$

where

$$F(x, \dot{x}) = \begin{pmatrix} \dot{x}_1 - x_2 \\ \dot{x}_2^3 - x_2\dot{x}_2 + x_1 \end{pmatrix}$$

Explicit and Implicit Solutions

Sometimes one can find explicit or implicit solutions to ODEs. For example, the separable first order initial value problem

$$(1) \qquad \frac{dx}{dt} = \frac{x}{\mu + x} \qquad x(0) = x_0$$

has the implicit solution

$$(2) \qquad t - (x - x_0) - \mu \ln\left(\frac{x}{x_0}\right) = 0$$

Despite this, the solution in (2) is hard to interpret let alone use to describe the solution $x(t)$.

Such observations lead to trying to uncover qualitative techiques for the examination of equations like (1).

Maps (Difference Eqns)

Discrete time dynamical systems.
Some examples:

$x_{n+1} = k\, x_n$ linear first order

$x_{n+2} + x_n = n$ linear 2nd order nonautomous

$x_{n+1} = k\, x_n^2$ nonlinear first order

One can also have systems:

$$x_{n+1} = 2x_n + y_n$$
$$y_{n+1} = 3x_n - y_n$$

2nd order linear system

Trajectories (orbits) of maps are sequences.

EXAMPLE $x_{n+1} = 2x_n \qquad x_0 = 1$

$$x_n = 2^n$$

with orbit $\{1, 2, 2^2, \ldots\}$

EXAMPLE $x_{n+2} - 5x_{n+1} + 6x_n = 0$

Letting $x_n = \lambda^n$ one finds $\lambda = 2, 3$ so

$$x_n = a\, 2^n + b\, 3^n$$

Partial Differential Eqns (an example)

Heat equation is a dynamical system.
The solution of

(1) $u_t = u_{xx}$ PDE

(2) $u(0,t) = u(\pi, t) = 0$ BC

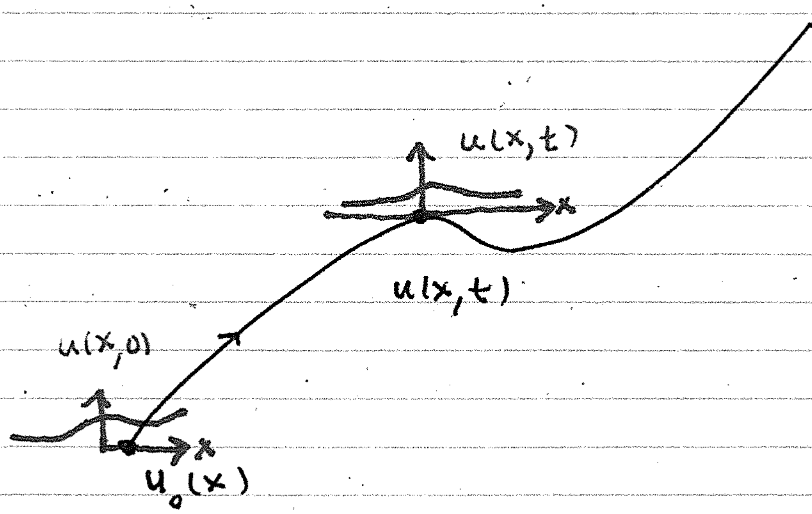
(3) $u(x,0) = \sin x$ IC

can be found using separation of variables

$$u(x,t) = e^{-t} \sin x$$

Of course this would be more complicated for arbitrary initial conditions

$$u(x,0) = u_0(x)$$



trajectory in function space