First order differential equations

$$\dot{x} = f(x, \mu)$$

where $x(t) \in \mathbb{R}$ and μ is a parameter(s). Issues include

(i) existence and uniqueness

(ii) equilibria and stability

(iii) solution dependence on μ

Defn: $\bar{x}$ is a fixed point of $\dot{x} = f(x)$ if $f(\bar{x}) = 0$

Remarks:

(1) $x(t) = \bar{x}$ is a solution of the IVP

$$\dot{x} = f(x) \qquad x(0) = \bar{x}$$

for all $t \geq 0$.

(2) Terminologies vary. The following are equivalent

fixed point

equilibria

steady state

invariant point

EX $\bar{x} = 1, -2$ are fixed points of

$$\dot{x} = x^2 + x - 2 = (x+2)(x-1)$$

EX $\bar{x} = 0$ is the sole equilibria of

$$\dot{x} = |x|$$

EX The ODE

$$\dot{x} = x^2 + 1$$

has <u>no</u> fixed points

EX Prove

$$\dot{x} = e^x - \tfrac{1}{2}x^2 = f(x)$$

has a unique fixed point.

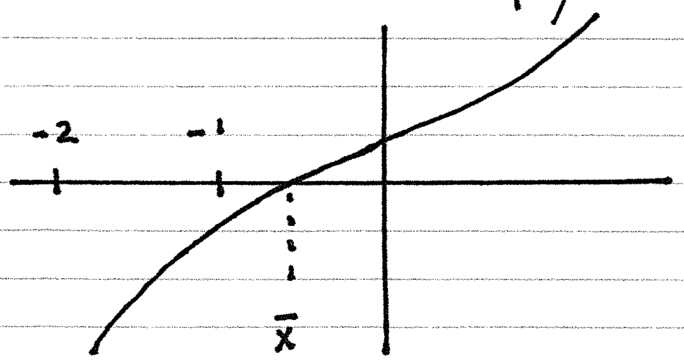
We appeal, in part, to the intermediate value theorem.

$$f(-1) = e^{-1} - \tfrac{1}{2} < 0$$

$$f(0) = 1 > 0$$

(monotone) $\quad f'(x) = e^x - x > 0 \quad$ Why?

These three facts imply the graph of $f(x)$ is:



In all that follows, $x(t)$ is a solution of

$$(1) \qquad \dot{x} = f(x) \qquad x(0) = x_0$$

and $\bar{x}$ is a fixed point of $\dot{x} = f(x)$.

<u>Defn</u> $\bar{x}$ is <u>stable</u> or <u>Lyapunov stable</u> if $\forall \varepsilon > 0$ there is a $\delta > 0$ s.t.

$$|x_0 - \bar{x}| < \delta \quad \Rightarrow \quad |x(t) - \bar{x}| < \varepsilon \text{ for all } t \geq 0$$

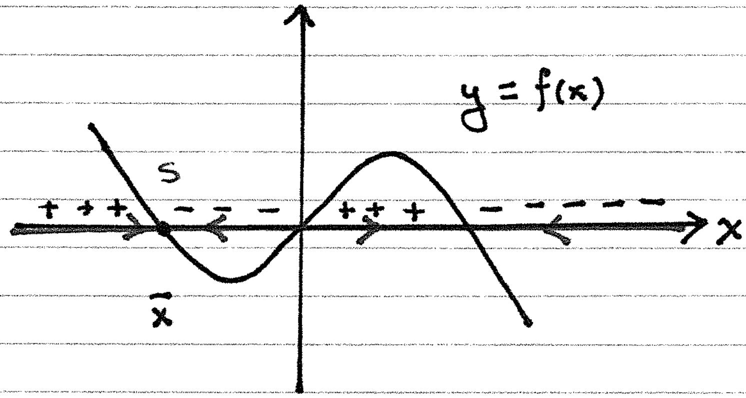
<u>Defn</u> $\bar{x}$ is <u>unstable</u> if it is not stable

<u>Defn</u> $\bar{x}$ is <u>asymptotically stable</u> if it is stable and $\exists \delta > 0$ such that

$$|x_0 - \bar{x}| < \delta \quad \Rightarrow \quad x(t) \to \bar{x} \text{ as } t \to \infty$$

To understand the subtelties of these definitions we will resort to examining phase portraits of $\dot{x} = f(x)$. If $f(x) > 0$ then $x(t)$ must be an increasing function of t. If $f(x) < 0$ then $\dot{x} < 0 \Rightarrow x(t)$ is decreasing. At fixed points $\bar{x}$, $f(\bar{x}) = 0$ so $x(t)$ does not change.

Phase Portrait of $\dot{x} = f(x)$ for $f(x)$ continuous

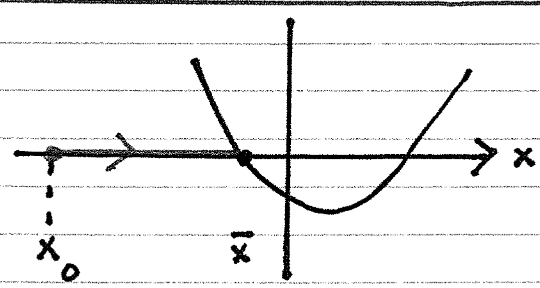


s = stable.

when $f(x) > 0$ then $\dot{x} > 0 \Rightarrow x(t)$ increasing.
when $f(x) < 0$ then $\dot{x} < 0 \Rightarrow x(t)$ decreasing.

Figure above (phase portrait) shows where $x(t)$ is $\uparrow$ and $\downarrow$. Also, the indicated fixed point $\bar{x}$ is stable.

A technical issue (near fixed points)



Consider

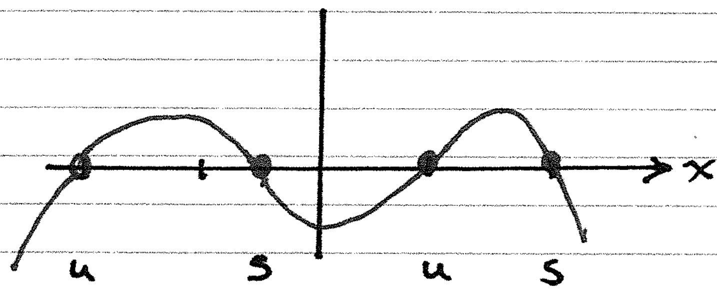
$\dot{x} = f(x) \qquad x(0) = x_0 < \bar{x}$

Is $\bar{x}$ asymptotically stable? Well, above for $x_0 < \bar{x}$ we know $x(t)$ increases monotonically and is bounded above. By properties of real numbers $\exists\, \bar{X}$ s.t. $x(t) \to \bar{X}$. Claim $\bar{X} = \bar{x}$. If not $\dot{x} \geq f(\bar{X}) > \delta > 0 \Rightarrow x(t) > \bar{X}$ for some t. Contradiction. Similar for $x_0 > \bar{x}$ ....

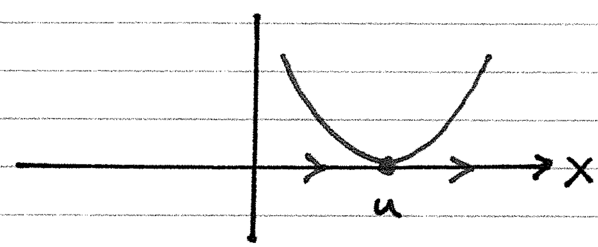
$\bar{x}$ is asymptotically stable.

EX $\qquad \dot{x} = (2x+1)(1-x)(x^2-4)$

Fixed points $\bar{x} = -\frac{1}{2}, 1, \pm 2$.

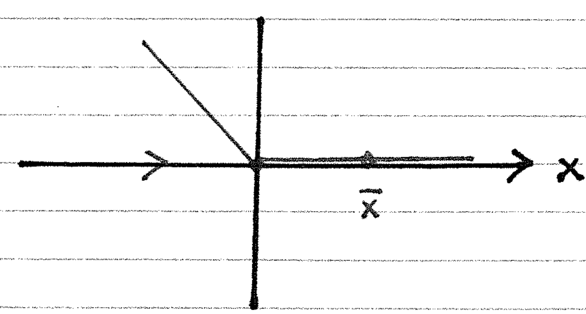


EX $\qquad \dot{x} = (x-1)^2 \qquad\qquad \bar{x} = 1$ sole.



unstable.

EX $\qquad \dot{x} = f(x) = \begin{cases} -x & x \le 0 \\ 0 & x > 0 \end{cases}$



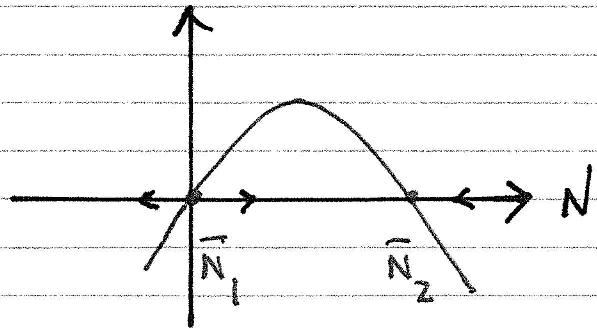
$\bar{x} \in [0,\infty)$ are all fixed points

All are stable.

None are asymptotically stable.

EX Population model (Logistic eqn)

$$\dot{N} = rN\left(1 - \frac{N}{K}\right) = f(N) \qquad (r, K \in \mathbb{R}^+)$$

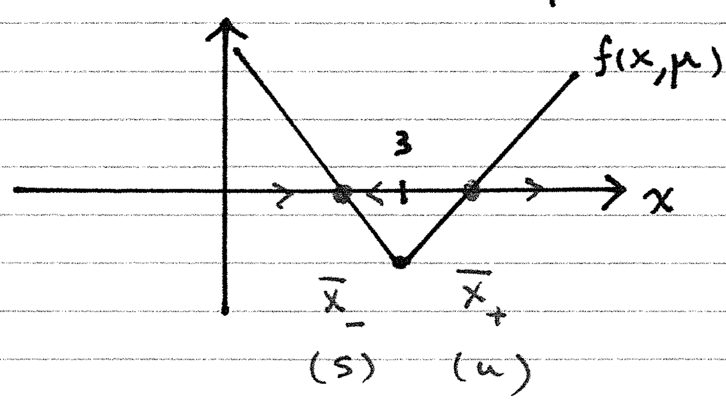


Here only $\bar{N}_2 = K$ is stable.

EX Bifurcation problem

$$\dot{x} = f(x,\mu) \equiv |x - 3| + \mu$$

has no fixed points if $\mu > 0$. If $\mu = 0$ then $\bar{x} = 3$ is sole fixed point. For $\mu < 0$



$\mu < 0$

One can show

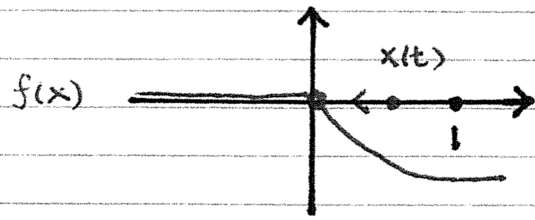
$$\bar{x}_-(\mu) = 3 + \mu \qquad \text{stable}$$

$$\bar{x}_+(\mu) = 3 - \mu \qquad \text{unstable.}$$

How long does it take to get to a fixed point?

EXAMPLE

$$\dot{x} = f(x) = \begin{cases} -\sqrt{x} & x > 0 \\ 0 & x \le 0 \end{cases} \qquad \underline{x(0) = 1}$$



Solve (separable)

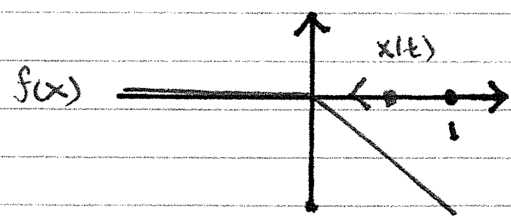
$$\int_1^x \frac{dx}{\sqrt{x}} = -\int_0^t dt$$

$$x(t) = \left(1 - \tfrac{1}{2}t\right)^2 \qquad \text{solution of IVP}$$

arrives at $\bar{x} = 0$ in <u>finite time</u> $t = 2$, i.e. $x(2) = 0$.

EXAMPLE

$$\dot{x} = f(x) = \begin{cases} -x & x > 0 \\ 0 & x \le 0 \end{cases} \qquad \underline{x(0) = 1}$$



Solve $\dot{x} = -x$ linear

$$x(t) = ce^{-t}$$

$$x(t) = e^{-t} \qquad \text{solution of IVP}$$

never arrives at $\bar{x} = 0$ but

$$\lim_{t \to \infty} x(t) = \bar{x} \qquad \text{"infinite" time}$$