2 Review of Linear Algebra

Here we give an overview of some linear algebra tools and definitions needed to solve and analyze the dynamics of linear systems in the next section.

First, the imaginary number i is that number such that $i^2 = -1$ and the symbol $\mathbb{C}$ will be used to denote the space of complex numbers. Thus $z \in \mathbb{C}$ has the form

$$z = a + ib \quad , \quad a, b \in \mathbb{R}$$

We will also need the identity

$$e^z = e^a(\cos\, b + i \sin\, b)$$

which is proveable using Taylor series expansions.

For any matrix $A \in \mathbb{C}^{n\times n}$, we define the nullspace $N(A)$ of A as:

$$N(A) = \{x \in \mathbb{C}^n : Ax = 0\}\,.$$

The zero vector is always in $N(A)$. A necessary condition for $N(A)$ to be nontrivial (not only the zero vector) is that its determinant vanishes. In this case, A is not invertible.

If $A \in \mathbb{C}^{2\times 2}$,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad , \quad a_{ij} \in \mathbb{C},$$

then its determinant is

$$det(A) = a_{11}a_{22} - a_{12}a_{21}$$

If $det(A) \neq 0$ the inverse matrix A^{-1} of $A \in \mathbb{C}^{2\times 2}$ exists and is given by the simple formula:

$$A^{-1} = \frac{1}{det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}\,.$$

For $A \in \mathbb{C}^{2\times 2}$ ("2 by 2" matrices) the computation of the nullspace is very simple. If $det(A) = 0$ then the (two) row vectors of A are necessarily dependent so row reduction is not needed to find a spanning vector for $N(A)$. The example below illustrates the determination of such a spanning vector.

Example 1 *For the matrix*

$$A = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix},$$

the equation $Ax = 0$ is equivalent to

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ -6x_1 + 3x_2 &= 0 \end{aligned}$$

Setting $x_1 = 1$ one finds $x_2 = 2$ so that by defining $\vec{\zeta} = (1,2)^T$, the nullspace of A is any multiple of $\vec{\zeta}$ or

$$N(A) = span\{\vec{\zeta}\}.$$

The nullspace is the line $x_2 = 2x_1$ in $\mathbb{R}^2$.

In order to solve linear systems of differential equations we must first define eigenvalues, eigenvectors and eigenspaces. Though we will only be dealing with real matrices A we will state the definitions as if A were complex.

Definition 10 *Let $A \in \mathbb{C}^{n \times n}$. A number $\lambda \in \mathbb{C}$ is an eigenvalue of A if there exists an $x \in \mathbb{C}^n$, $x \neq 0$ such that $Ax = \lambda x$. Any such x associated with an eigenvalue λ is an eigenvector of A. Further, for any eigenvalue λ of A we define the eigenspace $E_\lambda(A)$ of A as:*

$$E_\lambda(A) = \{x \in \mathbb{C}^n : Ax = \lambda x\}.$$

An alternate way of thinking of eigenvalues is that they are those λ for which $N(A - \lambda I)$ is nontrivial (here I is the identity matrix). This is only possible if $det(A - \lambda I)$ vanishes. Thus, eigenvalues are roots of the characteristic polynomial

$$P(\lambda) \equiv det(A - \lambda I) = 0 \,.$$

Example 2 *Let*

$$A = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} .$$

Then

$$P(\lambda) = det\left(\begin{bmatrix} -\lambda & 1 \\ 6 & 1-\lambda \end{bmatrix}\right) = \lambda^2 - \lambda - 6 = (\lambda - 3)(\lambda + 2) \,.$$

Thus, the eigenvalues of A are $\lambda_1 = 3$ and $\lambda_2 = -2$. To find an eigenvector $\vec{\zeta}_1$ of A associated with the eigenvalue λ_1 note that

$$A - \lambda_1 I = \begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix}$$

so that $\vec{\zeta}_1 = (1,3)^T$ and $E_{\lambda_1}(A) = span\{(1,3)^T\}$. In $\mathbb{R}^2$ this space is the line $x_2 = 3x_1$. For the other eigenvalue λ_2 one finds $\vec{\zeta}_2 = (1,-2)^T$ and the eigenspace $E_{\lambda_2}(A) = span\{(1,-2)^T\}$ is the line $x_2 = -2x_1$.

Now we have the standard diagonalization theorem:

Theorem 1 *(Diagonalization) If $A \in \mathbb{C}^{n \times n}$ has n distinct (nonequal) eigenvalues $\lambda_1, \lambda_2, \dots \lambda_n$ with associated eigenvectors $\vec{\zeta}_1, \vec{\zeta}_2, \dots \vec{\zeta}_n$ then the matrix S with $\vec{\zeta}_j$ as its columns*

$$S = [\vec{\zeta}_1, \vec{\zeta}_2, \dots \vec{\zeta}_n]$$

diagonalizes A as follows

$$S^{-1}AS = \Lambda$$

where

$$\Lambda = diag(\lambda_1, \lambda_2, \dots \lambda_n) = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix} .$$

It can easily be verified that in our previous example the matrix

$$S = [\vec{\zeta}_1, \vec{\zeta}_2] = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}$$

diagonalizes A. Note, however, that not all matrices can be diagonalized by their eigenvectors. Indeed, the matrix

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

has a characteristic polynomial

$$P(\lambda) = \lambda^2$$

and thus has only one eigenvalue $\lambda = 0$. Since $\lambda = 0$ is a double root it is said to have algebraic multiplicity 2. Noting, $E_0(A) = N(A)$ we see that the eigenspace is one dimensional and is spanned by the eigenvector $\vec{\zeta} = (1,0)^T$. Thus, in this instance there are not two eigenvectors with which to form S!

Eigenvectors can be normalized so their Euclidean norm is 1. For our previous example,

$$\| \vec{\zeta}_1 \| = \sqrt{10}$$

so that

$$\hat{\zeta}_1 = \left(\frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right)^T$$

is the normalized eigenvector associated with λ_1. Then, a symmetric [1] matrix $A \in \mathbb{R}^{n \times n}$ can be orthogonally diagonalized by an orthogonal matrix. If A is a real symmetric matrix satisfying the assumptions of the previuos Theorem then this orthogonal matrix Q is formed by the normalized eigenvectors:

$$Q = [\hat{\zeta}_1, \hat{\zeta}_2, \ldots \hat{\zeta}_n] \quad , \quad Q^{-1} A Q = \Lambda.$$

By defintion, a matrix is orthogonal if

$$Q^{-1} = Q^T$$

or that their transpose is their inverse. As a consequence

$$\| Qx \| = \| x \| \quad , \quad \forall x \in \mathbb{R}^n$$

or that they preserve length.

[1] $A^T = A$