Linear Planar systems

$$\dot{x}_1 = a_{11} x_1 + a_{12} x_2$$

$$\dot{x}_2 = a_{21} x_1 + a_{22} x_2$$

can be written

(1) $$\dot{x} = f(x) = Ax \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The system (1) is said to be linear since if y and z are any two solutions then so is any linear combination.

(2) $$x = c_1 y + c_2 z \qquad c_K \in \mathbb{R}$$

Then

$$\dot{x} = c_1 \dot{y} + c_2 \dot{z}$$

since y, z solns of (1)

$$\dot{x} = c_1 Ay + c_2 Az$$

$$\dot{x} = A(c_1 y + c_2 z)$$

$$\dot{x} = Ax$$

showing (2) is also a soln for any c_K.

Analytic Solutions and Fundamental Matrices

To solve the initial value problem

(1) $$\dot{x} = Ax \qquad x(0) = x_0$$

one must find two independent solution of $\dot{x} = Ax$. Finding them depends on the eigenvalues and eigenspaces of A. Suppose we find two such solutions $p_1(t)$ and $p_2(t)$. Then

$$\psi(t) = \begin{bmatrix} | & | \\ p_1(t) & p_2(t) \\ | & | \end{bmatrix}$$ Fundamental Matrix

and the general solution of (1) is

(2) $$x(t) = \psi(t)\, c \qquad c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Alternately (2) can be written

(3) $$x(t) = c_1 p_1(t) + c_2 p_2(t)$$

The values of c_k depend on the initial cond. At $t=0$, eqn (2) becomes

$$x_0 = \psi(0)\, c \quad \Rightarrow \quad c = \psi(0)^{-1} x_0$$

hence

(4) $$x(t) = \psi(t)\, \psi(0)^{-1} x_0$$

EXAMPLE Real Distinct eigenvalues $\lambda_1 < 0 < \lambda_2$

(1) $\dot{x} = Ax$ $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

Find eigenvalues

$$P(\lambda) = \det(A - \lambda I) = (1-\lambda)^2 - 4 = 0$$

Hence $\lambda_1 = -1$ and $\lambda_2 = 3$.

Can find eigenvectors for each eval.

$$(A - \lambda_1 I)\vec{\xi}_1 = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \vec{\xi}_1 = \vec{0} \qquad \vec{\xi}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$(A - \lambda_2 I)\vec{\xi}_2 = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} \vec{\xi}_2 = \vec{0} \qquad \vec{\xi}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Thus

$$\rho_1(t) = e^{-t}\vec{\xi}_1 \qquad \rho_2(t) = e^{3t}\vec{\xi}_2$$

General Soln

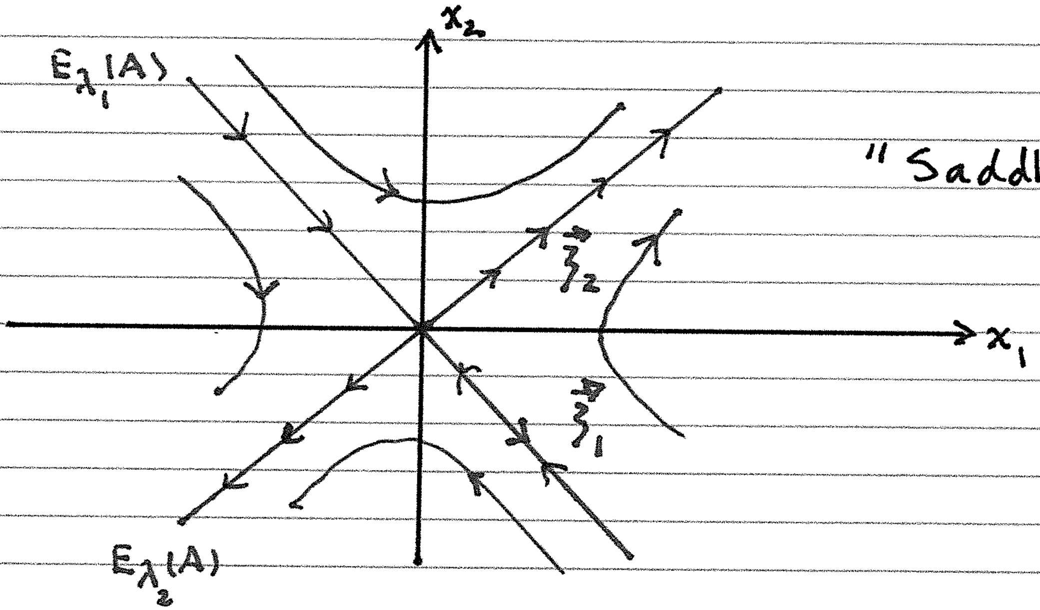
$$x(t) = c_1 e^{-t}\vec{\xi}_1 + c_2 e^{3t}\vec{\xi}_2$$

Real question is what do solutions look like in the plane.

Recall

$$\vec{x}(t) = \underset{\substack{\uparrow \\ \text{decays}}}{c_1 e^{-t}} \vec{\zeta}_1 + \underset{\substack{\uparrow \\ \text{grows}}}{c_2 e^{3t}} \vec{\zeta}_2$$

Can deduce a phase portrait of $\dot{x} = Ax$



"Saddle"

Eigenspaces $E_{\lambda_k}(A)$ are "stable" and "unstable"

Also, the sole equilibria $(0,0)$ is unstable

$$\boxed{\dot{x} = Ax \qquad A = \begin{bmatrix} a & 0 \\ 0 & -1 \end{bmatrix}}$$

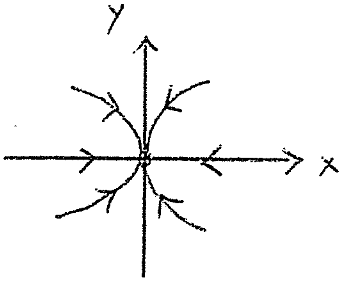
EXAMPLE $\quad \dot{x} = ax \quad (\bar{x}, \bar{y}) = (0,0)$ sole fx pt. if $a \neq 0$.

$\dot{y} = -y$

Solution for $x(0) = x_0$, $y(0) = y_0$ is

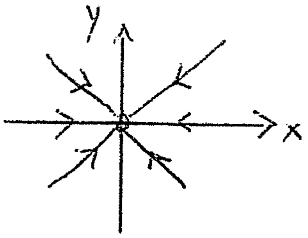
$$x(t) = x_0 e^{at} \qquad y(t) = y_0 e^{-t}$$

$a < -1$



isolated, a.stable NODE

$a = -1$



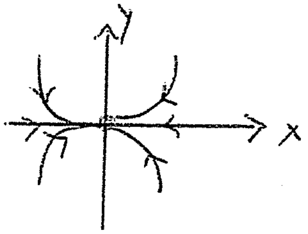
$\frac{y(t)}{x(t)} = K = \frac{y_0}{x_0}$, $\forall t \in \mathbb{R}$

so long as $x(t) \neq 0$.

isolated, a.s.

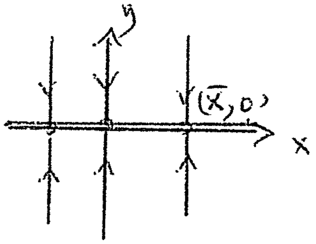
NODE (Star)

$a \in (-1, 0)$



isolated, a.stable NODE

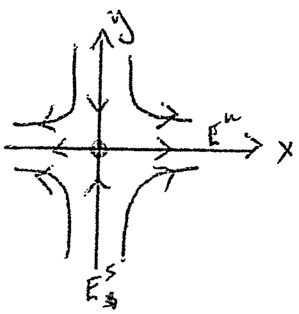
$a = 0$



non isolated. $y = 0$, $x \in \mathbb{R}$ line of fixed pts.

NOT ATTRACTING, IS LIAPUNOV $\Rightarrow$ Neutral stable

$a > 0$



UNSTABLE. SADDLE

(Stable and unstable manifolds)

Real Distinct eigenvalues $\lambda_1 \neq \lambda_2 \in \mathbb{R}$

The general solution of (1) is

$$\vec{x} = c_1 e^{\lambda_1 t} \underset{\substack{\uparrow}}{\vec{\xi}_1} + c_2 e^{\lambda_2 t} \underset{\substack{\uparrow}}{\vec{\xi}_2}$$

eigenvectors for λ_1, λ_2

EXAMPLE

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix} \vec{x} = A\vec{x}$$

$$P(\lambda) = \det(A - \lambda I) = (\lambda - 3)(\lambda - 4)$$

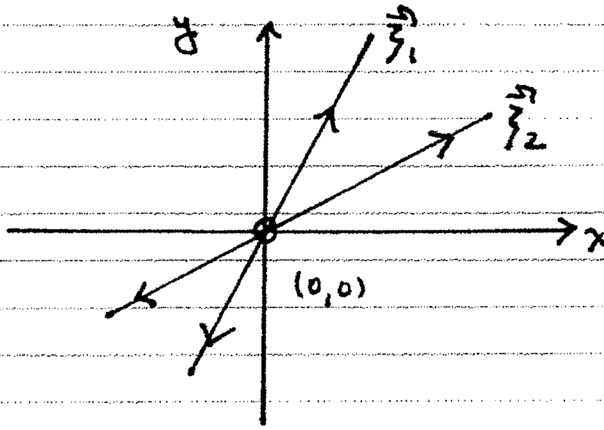
after some calculations.

$\lambda_1 = 3$ $\quad (A - \lambda_1 I)\vec{\xi} = \begin{bmatrix} 3 & -3 \\ 2 & -2 \end{bmatrix}\vec{\xi} = \vec{0}$ $\quad \vec{\xi}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 4$ $\quad (A - \lambda_2 I)\vec{\xi} = \begin{bmatrix} 2 & -3 \\ * & * \end{bmatrix}\vec{\xi} = \vec{0}$ $\quad \vec{\xi}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

General solution

$$\vec{x}(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 3 \\ 2 \end{pmatrix}$$ grows.



Unstable Node

EXAMPLE $\boxed{\text{unstable node} \qquad 0 < \lambda_1 < \lambda_2}$

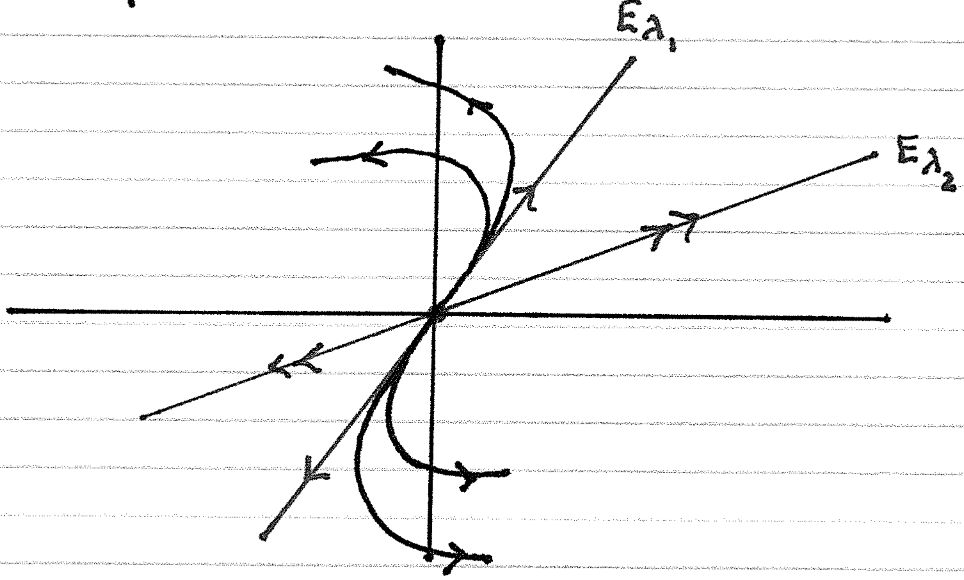
$$\dot{x} = Ax \qquad A = \begin{bmatrix} 6 & -3 \\ 2 & 1 \end{bmatrix}$$

Calculations show evals, evects, and solutions are:

$\lambda_1 = 3 \qquad \vec{\xi}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad p_1(t) = e^{3t}\vec{\xi}_1$ (weaker)

$\lambda_2 = 4 \qquad \vec{\xi}_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \qquad p_2(t) = e^{4t}\vec{\xi}_2$

Phase portrait



(0,0) is an unstable node

Remark If one negates A then $\lambda_1 = -3$, $\lambda_2 = -4$ and (0,0) is a stable node with the direction of the trajectories opposite to above.

EXAMPLE **(Saddle)** $\lambda_2 < 0 < \lambda_1$

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x} = A\vec{x}$$

Here the characteristic polynomial is

$$P(\lambda) = \lambda^2 - 1 \qquad \lambda = \pm 1$$

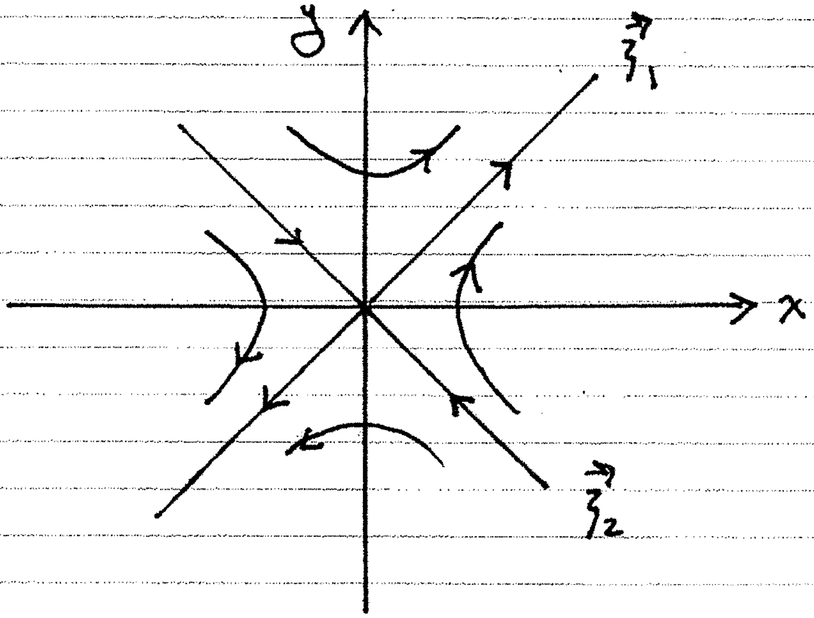
It is easily verified we have eigenvectors:

$$\lambda_1 = +1 \qquad (A - \lambda_1 I)\vec{\xi} = \begin{bmatrix} -1 & 1 \\ * & * \end{bmatrix}\vec{\xi} \qquad \vec{\xi}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = -1 \qquad (A - \lambda_2 I)\vec{\xi} = \begin{bmatrix} 1 & 1 \\ * & * \end{bmatrix}\vec{\xi} \qquad \vec{\xi}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

General Solution

$$\vec{x}(t) = \overbrace{c_1 e^{\lambda_1 t}}^{\text{grows}} \vec{\xi}_1 + \overbrace{c_2 e^{\lambda_2 t}}^{\text{decays}} \vec{\xi}_2$$



EXAMPLE $\boxed{\text{Saddle} \quad \lambda_1 < 0 < \lambda_2}$

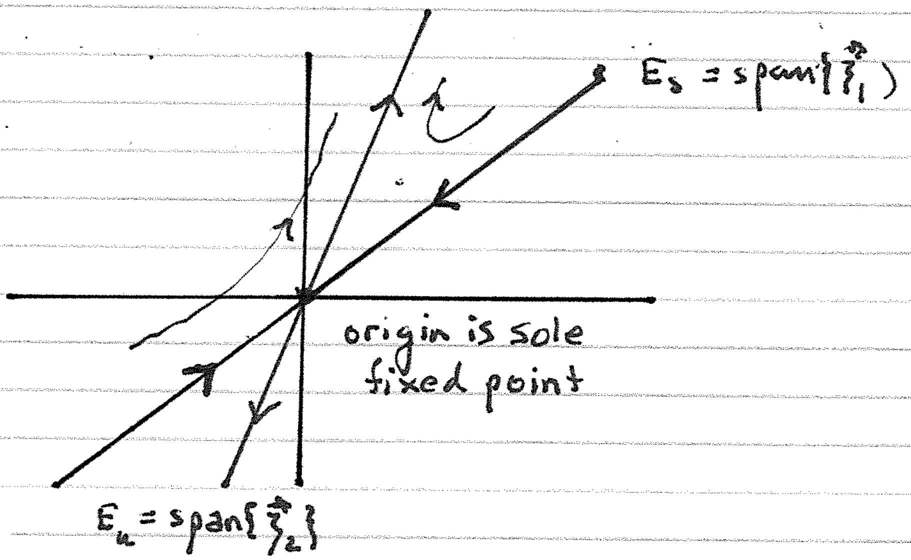
$$\dot{x} = Ax \qquad A = \begin{bmatrix} -2 & 1 \\ -5 & 4 \end{bmatrix}$$

Calculations show evals, evects and solns are

$$\lambda_1 = -1 \qquad \vec{\xi}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \rho_1(t) = e^{-t}\vec{\xi}_1$$

$$\lambda_2 = 3 \qquad \vec{\xi}_2 = \begin{pmatrix} 1 \\ 5 \end{pmatrix} \qquad \rho_2(t) = e^{3t}\vec{\xi}_2$$

General solution $x(t) = c_1\rho_1(t) + c_2\rho_2(t)$



(0,0) is an unstable saddle.

Complex eigenvalues $\lambda = \alpha + i\beta \in \mathbb{C}$

Eigenvalues and eigenvectors are both complex. The complex solution is

$$\vec{x}(t) = e^{\lambda t}\vec{\xi} = \vec{x}_r(t) + i\,\vec{x}_i(t)$$

Each of the real and imaginary parts of $\vec{x}$ are independent solutions to

$$\frac{d\vec{x}}{dt} = A\vec{x}$$

After computing $\vec{\xi} = \vec{a} + i\vec{b}$ and expanding

$$\vec{x} = e^{(\alpha+i\beta)t}(\vec{a} + i\vec{b})$$

and using $e^{i\beta t} = \cos\beta t + i\sin\beta t$, two real independent solutions are

$$\vec{x}_1(t) = e^{\alpha t}(\cos\beta t\,\vec{a} - \sin\beta t\,\vec{b})$$

$$\vec{x}_2(t) = e^{\alpha t}(\sin\beta t\,\vec{a} + \cos\beta t\,\vec{b})$$

Note even though solns always oscillate

$\alpha > 0$ $\quad$ $\vec{x}_K$ grow

$\alpha < 0$ $\quad$ $\vec{x}_K \to 0$

$\alpha = 0$ $\quad$ $\vec{x}_K$ periodic

Here β is the frequency.

EXAMPLE Complex = Spiral

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x} = A\vec{x}$$

Characteristic Polynomial

$$P(\lambda) = \lambda^2 - 2\lambda + 2 = 0 \Rightarrow \lambda = \overset{\alpha + i\beta}{1 + i}$$

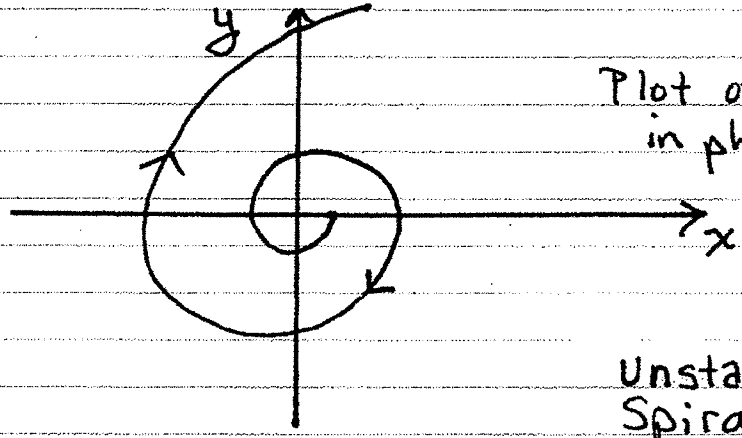
with associated eigenvector

$$\vec{\xi} = \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \vec{a} + i\vec{b}$$

Ultimately $\vec{x} = e^t(\cos t + i \sin t)(\vec{a} + i\vec{b})$ yields the general solution

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) = c_1 \underbrace{\begin{pmatrix} e^t \cos t \\ -e^t \sin t \end{pmatrix}}_{\vec{x}_1(t)} + c_2 \underbrace{\begin{pmatrix} e^t \sin t \\ e^t \cos t \end{pmatrix}}_{\vec{x}_2(t)}$$

Notice $\|x_1(t)\| = e^t$ so $\vec{x}_1(t) \to \infty$ as $t \to \infty$ but it does so by spiraling outward. Same with $\vec{x}_2$



Plot of $\vec{x}_1(t)$ in phase plane.

Unstable Spiral

EXAMPLE Simple Purely imaginary case $\lambda = i\beta$

$$\vec{x}' = A\vec{x} = \begin{bmatrix} 0 & -8 \\ 2 & 0 \end{bmatrix} \vec{x}$$

Characteistic Polynomial

$$P(\lambda) = \lambda^2 + 16 \qquad \lambda = 4i$$

Eigenvector

$$(A - \lambda I) = \begin{bmatrix} -4i & -8 \\ 2 & -4i \end{bmatrix} \qquad \vec{\xi} = \begin{pmatrix} 2i \\ 1 \end{pmatrix}$$

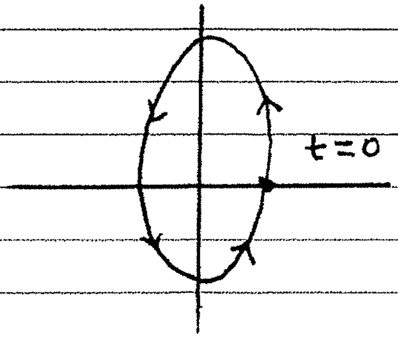
General solution

$$\vec{x}(t) = c_1 \begin{pmatrix} -2\sin 4t \\ \cos 4t \end{pmatrix} + c_2 \underbrace{\begin{pmatrix} 2\cos 4t \\ \sin(4t) \end{pmatrix}}_{\vec{x}_2(t)}$$

To see what these trajectories, look at $\vec{x}_2(t)$

$$\vec{x}_2 = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} 2\cos 4t \\ \sin 4t \end{pmatrix} \Rightarrow \left(\frac{x}{2}\right)^2 + y^2 = 1$$

is on ellipse



EXAMPLE Purely Imaginary $\lambda = i\beta$

$$\vec{x}\,' = A\vec{x} \qquad A = \begin{bmatrix} -2 & -4 \\ 10 & 2 \end{bmatrix}$$

Characteristic Polynomial (after calculations)

$$P(\lambda) = \lambda^2 + 36 = 0 \qquad \lambda = 6i \qquad (\alpha = 0)$$

Eigenvector

$$(A - \lambda I) = \begin{bmatrix} -2-6i & -4 \\ * & * \end{bmatrix} \qquad \vec{\xi} = \begin{pmatrix} 2 \\ -1-3i \end{pmatrix}$$

General solution

$$\vec{x}(t) = c_1 \underbrace{\begin{pmatrix} 2\cos 6t \\ -\cos 6t + 3\sin 6t \end{pmatrix}}_{\vec{x}_1(t)} + c_2 \begin{pmatrix} 2\sin 6t \\ -\sin 6t - 3\cos 6t \end{pmatrix}$$

Are ellipses with different axes. From $\vec{x}_1(t)$

$$\left(\frac{x}{2}\right)^2 + \left(\frac{x}{6} + \frac{y}{3}\right)^2 = 1$$

$$\cos^2 + \sin^2 = 1$$

Has axes:

$$x = 0 \qquad x + 2y = 0$$

EXAMPLE Line of fixed points

$$\dot{x} = Ax \qquad A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

Can easily find characteristic polynomial

$$P(\lambda) = \det(A - \lambda I) = \lambda^2 - 4\lambda$$

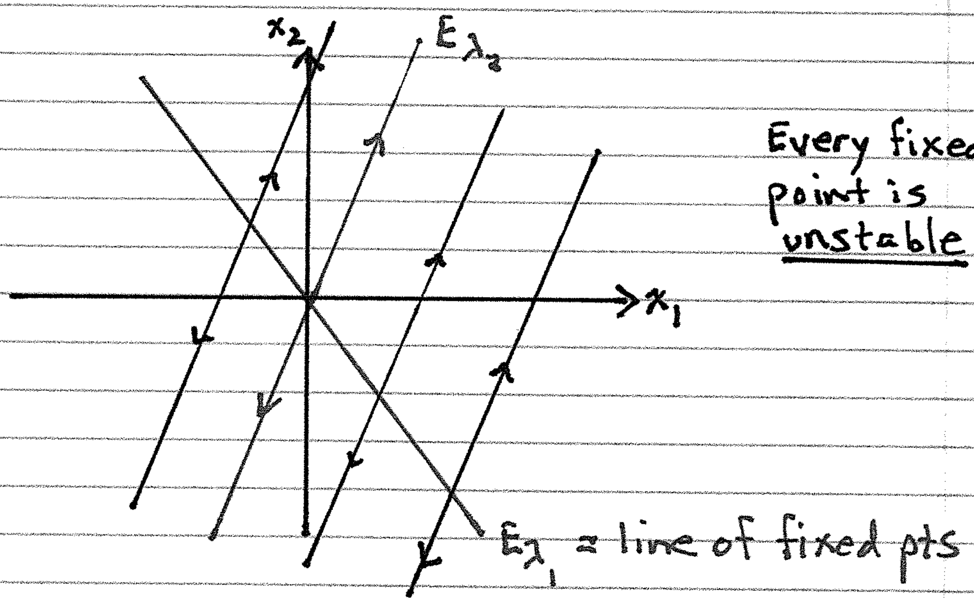
hence $\lambda_1 = 0$ and $\lambda_2 = 4$. Since $\lambda_1 = 0$ the nullspace $N(A)$ is nontrivial and its elements are all fixed points. Suppressing calculations one can easily find e-vectors:

$$\lambda_1 = 0 \qquad \vec{\xi}_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \qquad E_{\lambda_1} = \text{span}\{\vec{\xi}_1\}$$

$$\lambda_2 = 4 \qquad \vec{\xi}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad E_{\lambda_2} = \text{span}\{\vec{\xi}_2\}$$

General solution is

$$x(t) = c_1 \vec{\xi}_1 + c_2 e^{4t} \vec{\xi}_2$$



Every fixed point is <u>unstable</u>

Repeated roots

Are effectively no different than nodes but the general solutions are more complex. Suppose that the characteristic polynomial of

$$(1) \qquad \dot{x} = Ax \qquad A \in \mathbb{R}^{2\times 2}$$

$$P(\lambda) = (\lambda - \lambda_1)^2$$

It can be shown that if $\vec{\eta}_1$ and $\vec{\eta}_2$ are solns of

$$(A - \lambda_1 I)\vec{\eta}_1 = 0$$

$$(A - \lambda_1 I)\vec{\eta}_2 = \vec{\eta}_1$$

the general solution of (1) is

$$x(t) = c_1 e^{\lambda_1 t}\vec{\eta}_1 + c_2(t\vec{\eta}_1 + \vec{\eta}_2)e^{\lambda_1 t}$$

Stability can be ascertained by noting

$$\lambda_1 < 0 \quad \Rightarrow \quad x(t) \to 0 \quad \text{as } t \to \infty$$

$$\lambda_1 > 0 \quad \Rightarrow \quad x(t) \to \infty \quad \text{as } t \to \infty$$

Phase portraits are similar to nodes.

EXAMPLE

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \Rightarrow P(\lambda) = (\lambda - 2)^2 \qquad \lambda_1 = 2$$

Since $\lambda_1 > 0$ the origin is a unstable node