Linear systems classification

$$\boxed{\frac{d\vec{x}}{dt} = A\vec{x} \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}}$$

The stability of all fixed points (and the origin in particular) are determined by the eigenvalues of A. The characteristic polynomial is:

$$P(\lambda) = (a-\lambda)(d-\lambda) - bc$$

$$P(\lambda) = \lambda^2 - (a+d)\lambda + (ad-bc)$$

which can be written

(1) $$P(\lambda) = \lambda^2 - \mathrm{Tr}A\,\lambda + \det A$$

In equation (1)

$$\mathrm{Tr}A = a+d \qquad \text{"trace of A"}$$

$$\det A = ad - bc \qquad \text{"det of A"}$$

Hence the eigenvalues of A are roots of the quadratic in (1)

(2) $$\boxed{\lambda_{\pm} = \frac{1}{2}\left(\mathrm{Tr}A \pm \sqrt{\mathrm{Tr}A^2 - 4\det A}\right)}$$

From expression (2) we can deduce:

i) $\det A < 0$ — $\lambda_\pm$ real opposite signs
<u>saddle</u>

ii) $\det A > 0$, $\det A < \frac{1}{4} \mathrm{Tr} A^2$ — $\lambda_\pm$ real same sign (as $\mathrm{Tr} A$)
<u>node</u>

iii) $\det A > 0$, $\det A = \frac{1}{4} \mathrm{Tr} A^2$ — repeated real root
<u>node</u>

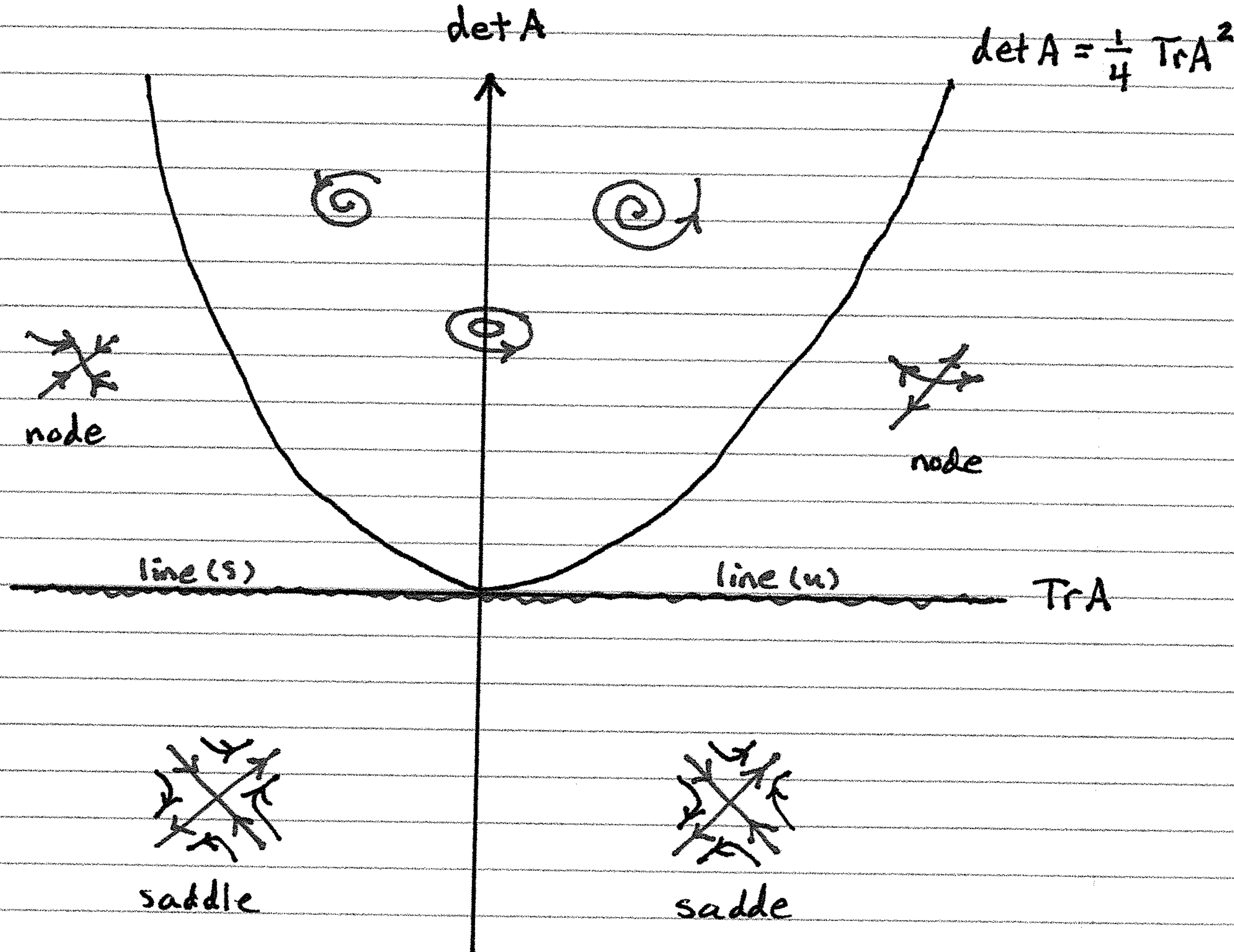
iv) $\det A > 0$, $\mathrm{Tr} A = 0$ — purely imaginary λ
<u>center</u>

v) $\det A > 0$, $\det A > \frac{1}{4} \mathrm{Tr} A^2$ — Complex eigenvalues λ with real part $\mathrm{Re}\,\lambda = \frac{\mathrm{Tr} A}{2}$
<u>spiral</u>

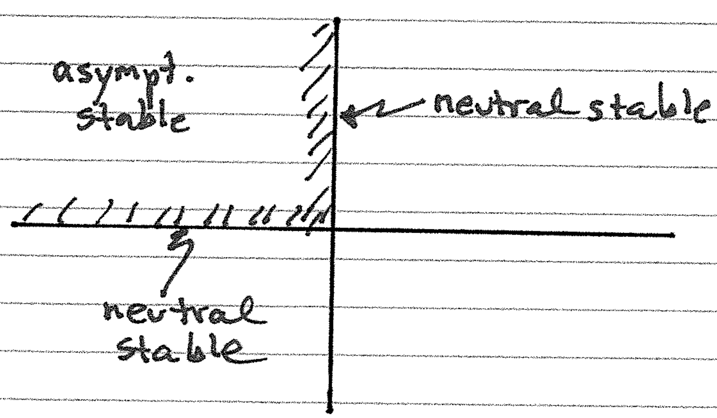
vi) $\det A = 0$ — line of equilibria
$\lambda_+ = \mathrm{Tr} A$ detemines stability

We summarize this on next page.

Classification of fixed point(s)



Note that only in quadrant II is (0,0) stable



Linear Manifolds

Suppose $(\lambda_k, \vec{\xi}_k)$ are the (complex) eigenvalue/eigenvector pairs of A

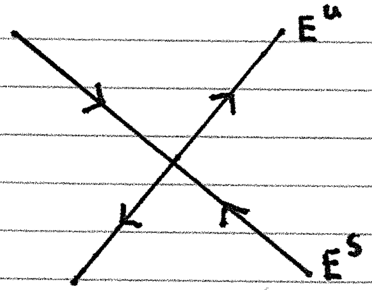
$$A\vec{\xi}_k = \lambda_k \vec{\xi}_k \qquad \vec{\xi}_k = \vec{x}_k + i\,\vec{y}_k$$

we define the following three special spaces

$$E^s(0) = \text{span}\{ x_k, y_k : \mathrm{Re}\,\lambda_k < 0 \}$$
$$E^c(0) = \text{span}\{ x_k, y_k : \mathrm{Re}\,\lambda_k = 0 \}$$
$$E^u(0) = \text{span}\{ x_k, y_k : \mathrm{Re}\,\lambda_k > 0 \}$$

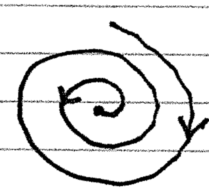
are the linear stable, center and unstable manifolds of $\bar{x} = (0,0)$.

EXAMPLE



saddle

no E^c



$E^s = \mathbb{R}^2$

stable spiral

no E^c

no E^u

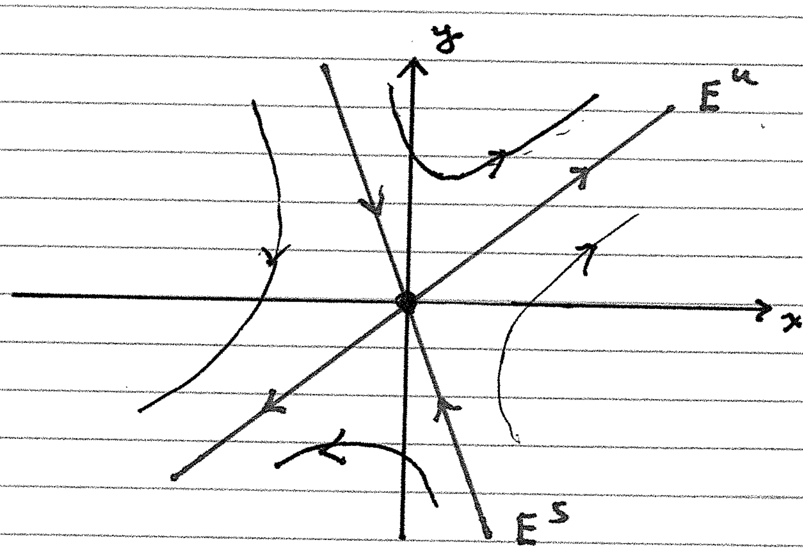
EXAMPLE $\quad A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$

Since $\det A < 0$ the origin is a center. Can find eigenvalue/eigenvector pairs

$$\lambda_+ = 2 \qquad \vec{\xi}_+ = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad E^u = \text{span}\{\vec{\xi}_+\}$$

$$\lambda_- = -2 \qquad \vec{\xi}_+ = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \qquad E^s = \text{span}\{\vec{\xi}_-\}$$

allow the qualitative sketch



unstable saddle

EXAMPLE Unstable node

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

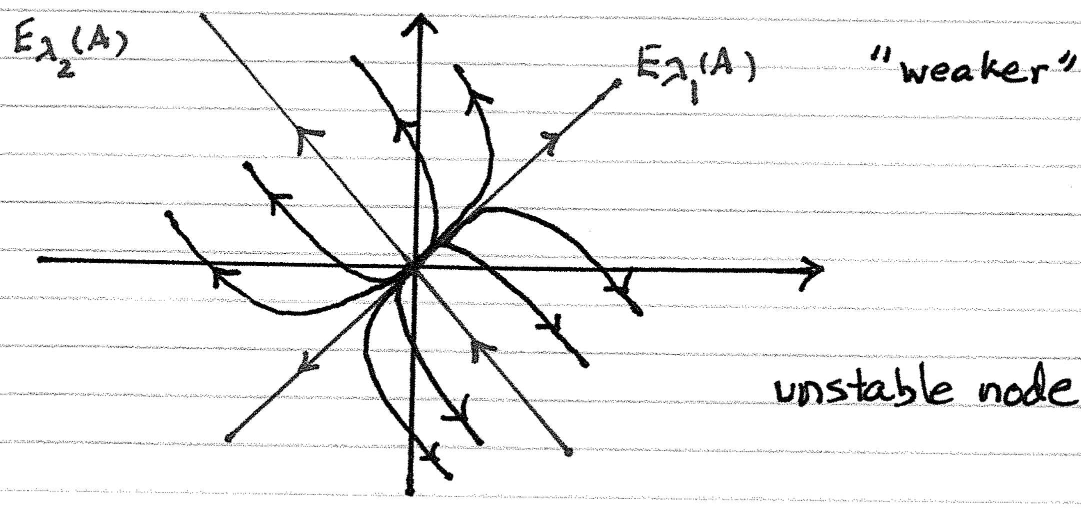
Here $\det A = 8$ and $\operatorname{Tr} A = 6$ hence

$\det A > 0$ and $\det A < \frac{1}{4} \operatorname{Tr} A^2 \Rightarrow$ node

Can easily find $(\lambda_K, \vec{\xi}_K)$

$\lambda_1 = 2 \qquad \vec{\xi}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = 4 \qquad \vec{\xi}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$



unstable node

$E^u(0) = \mathbb{R}^2$ (all of the plane)

$E^s(0) = \{0\}$

$E^c(0) = \{0\}$

EXAMPLE **Center**

$$A = \begin{bmatrix} 4 & -4 \\ 20 & -4 \end{bmatrix}$$

Complex eigenvalue/eigenvector pair

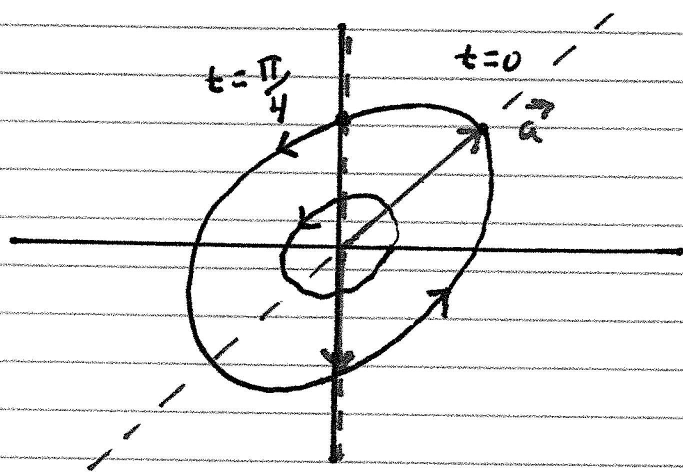
$$\lambda = 2i \qquad \vec{\xi} = \vec{a} + i\vec{b} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + i\begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

General solution is a linear combination of:

$$u(t) = \cos 2t\, \vec{a} - \sin 2t\, \vec{b} \qquad u(0) = \vec{a}$$

$$v(t) = \sin 2t\, \vec{a} + \cos 2t\, \vec{b} \qquad v(0) = \vec{b}$$

Carefully plot $u(t)$ below



$u(0) = \vec{a}$
$u(\frac{\pi}{4}) = -\vec{b}$
$u(\frac{\pi}{2}) = -\vec{a}$
$u(\frac{3\pi}{4}) = \vec{b}$

Neutrally stable

Center

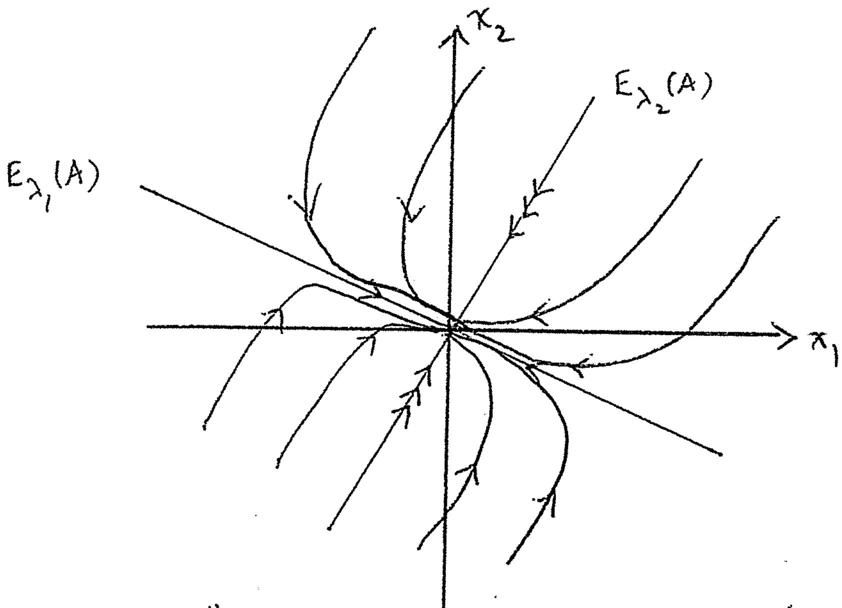
$$E^c(0) = \mathbb{R}^2$$

$$E^s(0) = \{\ \}$$

$$E^u(0) = \{\ \}$$

EX **STABLE NODE** $A = \begin{bmatrix} -13 & -27 \\ -9 & -31 \end{bmatrix}$ $\lambda_1 = -4 \quad \vec{\xi}_1 = (-3, 1)^T$

$\lambda_2 = -40 \quad \vec{\xi}_2 = (1, 1)^T$



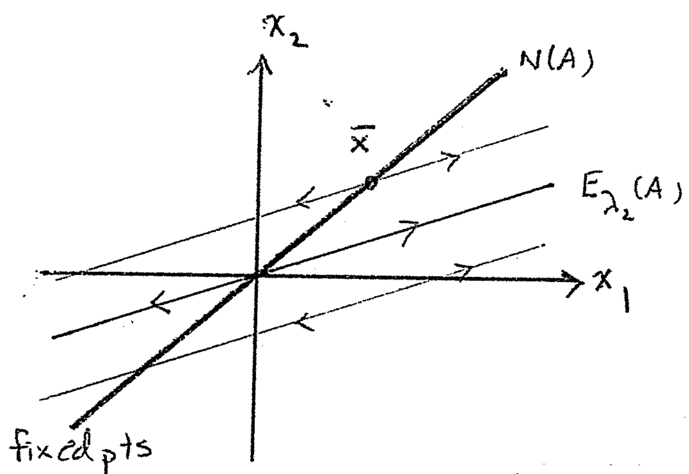
$x(t) = c_1 e^{-4t} \vec{\xi}_1 + c_2 e^{-40t} \vec{\xi}_2$

E

$E^s(0) = \mathbb{R}^2$
$E^u(0) = \{0\}$

"squashed onto $E_{\lambda_1}(A)$"

EX **DEGENERATE (LINE OF FIXED)** $A = \begin{bmatrix} 4 & -6 \\ 2 & -3 \end{bmatrix}$



$\lambda_1 = 0 \qquad \vec{\xi}_1 = \begin{pmatrix} 3/2 \\ 1 \end{pmatrix}$

$\lambda_2 = 1 \qquad \vec{\xi}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$x(t) = c_1 \vec{\xi}_1 + c_2 e^t \vec{\xi}_2$

Non isolated, Neutral stability.

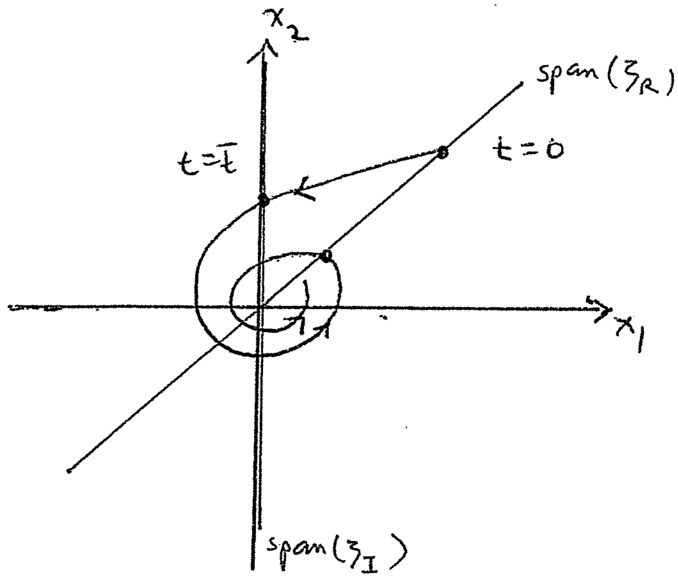
$E^s(\bar{x}) = \{\bar{x}\}$. However, for $\bar{x} = (\bar{x}_1, \bar{x}_2)$

$E^u(\bar{x}) = \{(x_1, x_2) : \quad x_2 - \bar{x}_2 = \frac{1}{2}(x_1 - \bar{x}_1)\}$

is a straight line. $E^s(\bar{x}) \cup E^u(\bar{x}) \neq \mathbb{R}^2$ in this case

EX STABLE SPIRAL

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -2 \end{bmatrix}$$



$$\lambda_{\pm} = -\frac{1}{2} \pm \frac{\sqrt{15}}{2} i$$

$$\xi = \begin{pmatrix} 4 \\ 3 \end{pmatrix} + i \begin{pmatrix} 0 \\ -\sqrt{15} \end{pmatrix} = \vec{\xi}_R + i\vec{\xi}_i$$

$\omega = \frac{\sqrt{15}}{2}$; NOT PERIODIC.

Choose I.C. so that

$$x(t) = u(t) = e^{-\frac{1}{2}t}\left(\cos \omega t \, \vec{\xi}_R - \sin \omega t \, \vec{\xi}_I\right)$$

Then, when $t = \bar{t} = \frac{\pi}{2\omega}$, $x(\bar{t}) = -e^{-\frac{1}{2}\bar{t}} \, \vec{\xi}_I$