1 Linear Planar Systems - Definition and Fixed Points

A linear system of differential equations in $\mathbb{R}^2$ is a system of the form:

$$\begin{aligned} \dot{x_1} &= a_{11}x_1 + a_{12}x_2 \\ \dot{x_2} &= a_{21}x_1 + a_{22}x_2 \end{aligned}$$

where $a_{i,j}, i, j = 1, 2$, can be functions of t. Unless otherwise stated, however, we will assume $a_{i,j}$ are constants.

Letting $x = (x_1, x_2)^T$ be a column vector with $x_1(t)$ and $x_2(t)$ as its components, the system above can be written:

$$\dot{x} = Ax \tag{1}$$

where the matrix $A \in \mathbb{R}^{2\times 2}$ is:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

Providing $det(A) \neq 0$ the system $\dot{x} = Ax$ has the sole fixed point $x = (0, 0)$. If $det(A) = 0$ then every $\bar{x} \in N(A)$ is a fixed point.

Example 1 *Let*

$$\dot{x} = Ax = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Since $det(A) = 0$ *the nullspace is nontrivial. The first row of* $Ax = 0$ *is equivalent to* $2x_1 - x_2 = 0$. *Setting* $x_1 = 1$ *yields* $x_2 = 2$ *so that* $N(A)$ *is spanned by the vector* $\zeta = (1, 2)^T$, *i.e.,*

$$N(A) = span\{(1, 2)^T\}.$$

Geometrically, $N(A)$ *is the line* $x_2 = 2x_1$. *All points on this line are fixed points of* $\dot{x} = Ax$.

2 Solutions to IVP of Linear Systems in the Plane

Let $p_1(t)$ and $p_2(t)$ be two two solutions of the system $\dot{x} = Ax$, where $A \in \mathbb{R}^{2\times 2}$. That is,

$$\dot{p}_i = Ap_i \quad , \quad i = 1, 2$$

where $p_i(t) \in \mathbb{R}^2$. The Wronskian of p_1 and p_2 is the determinant of the matrix whose columns are formed by the column vector solutions p_i, i.e.,

$$W(p_1, p_2) = det[p_1|p_2].$$

The two solutions p_1 and p_2 are said to be linearly independent if their Wronskian does not vanish [1]. If the solutions are linear independent then the general solution of the initial value problem

$$\dot{x} = Ax \quad , \quad x(0) = x_0$$

[1] It can be shown [1] that the Wronskian of two solutions is either identically zero or it never vanishes

is given as

$$x(t) = c_1 p_1(t) + c_2 p_2(t)$$

where c_1 and c_2 are constants.

For any matrix A and vector $c = (c_1, c_2)^T$, the product Ac yields a linear combination of the columns of A. Thus, by defining the matrix

$$\Psi(t) = [p_1(t)|p_2(t)]$$

the general solution $x(t)$ can be written:

$$x(t) = \Psi(t)c\,.$$

Since the solutions forming the columns of Ψ are linearly independent, Ψ is invertible. Given the initial condition for x,

$$x(0) = x_0 = \Psi(0)c \quad \Rightarrow \quad c = \Psi(0)^{-1}x_0\,.$$

Therefore,

$$x(t) = \Phi(t)x_0 \quad , \quad \Phi(t) = \Psi(t)\Psi(0)^{-1}\,. \tag{2}$$

The matrix Φ is referred to as the <u>Fundamental Solution Matrix</u> for the problem although some authors also call Ψ a Fundamental Solution Matrix. There is only one Φ but the Ψ are not unique. For example one could have just as easily defined $\Psi = [ap_1|p_2]$ where a is any constant.

Since $\Phi(t)$ is unique it is sometimes written

$$\Phi(t) = e^{At}$$

where A is the original matrix defining the planar system. Precise definitions for functions of matrices ($e^A, sin(A), etc.$) is part of the subject of spectral theory (in Functional Analysis) and is usually accomplished using Taylor series. For instance, since A^n makes sense for any integer so the convergence of the series

$$e^{At} = I + tA + \frac{1}{2!}t^2A^2 + \cdots$$

can be discussed using matrix norms. It can be shown that the series on the right does converge to the fundamental matrix $\Phi(t)$ but we do not need to discuss such issues here. Be aware, however, that some books develop the theory for linear systems using a complete development of the definition of e^{At}.

Given a matrix A, equation (2) implies the solution $x(t)$ for any initial condition x_0 can be found if one can determine $\Psi(t)$. In most instances, this this amounts to finding the eigenvalues and eigenvectors of A. To see why this is, suppose one assumes a solution of the form

$$x(t) = e^{\lambda t}\zeta \quad , \quad \zeta = \begin{pmatrix} \zeta_1 \\ \zeta_2 \end{pmatrix}$$

where $\lambda \in \mathbb{C}$ is some constant and $\zeta \in \mathbb{C}^2$ is some constant vector. Substituting this expression into $\dot{x} = Ax$ yields

$$\lambda e^{\lambda t}\zeta = e^{\lambda t}A\zeta$$

or

$$A\zeta = \lambda\zeta \, .$$

If λ were chosen so that the only ζ which solved this problem were $\zeta = 0$ then the resulting solution $x(t) \equiv 0$ is uninteresting. However, if λ is an eigenvalue of A then there do exist nontrivial $\zeta \in N(A - \lambda I)$. Therefore, it appears that a prerequisite for determining $\Psi(t)$ is to find all the eigenvalues and eigenvectors of A. Although this is true, some other issues complicate matters but overall the construction of the Fundamental Solution Matrix can be categorized into three classes which we discuss in the subsequent three sections.

2.1 Real, Distinct Eigenvalues

Suppose that $A \in \mathbb{R}^{2\times 2}$ has two real and distinct eigenvalues λ_1, λ_2, $\lambda_1 \neq \lambda_2$ with respective eigenvectors ζ_1 and ζ_2 (Note here that ζ_i are vectors and not components of a vector). From basic linear algebra theory it can be shown that these eigenvectors are independent and that as a result the following two solutions are linearly independent [2]

$$x_1(t) = e^{\lambda_1 t}\zeta_1 \quad , \quad x_2(t) = e^{\lambda_2 t}\zeta_2$$

Thus, a Fundamental Solution Matrix is:

$$\Psi(t) = [e^{\lambda_1 t}\zeta_1 | e^{\lambda_2 t}\zeta_2] \, ,$$

from which $\Phi(t)$ can be computed.

Example 2 *Let*

$$\dot{x} = Ax = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic polynomial for A is

$$P(\lambda) = det(A - \lambda I) = (1 - \lambda)^2 - 4$$

The roots of P are the eigenvalues. In this case $\lambda_1 = 3$ *and* $\lambda_2 = -1$.

$$(A - \lambda_1 I) = \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix}$$

so that $\zeta_1 = (1, 2)^T$ *is an eigenvector associated with eigenvalue* λ_1. *Similarly,* $\zeta_2 = (1, -2)^T$. *The two inedepndent solutions are*

$$x_1(t) = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} \quad , \quad x_2(t) = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix} \, .$$

A Fundamental Solution Matrix is

$$\Psi(t) = \begin{bmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{bmatrix} \, ,$$

[2] we omit the details

from which one finds

$$\Psi(0) = \begin{bmatrix} 1 & 1 \\ 2 & -2 \end{bmatrix} \quad , \quad \Psi(0)^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & -\frac{1}{4} \end{bmatrix} \quad .$$

To find the solution of the initial value problem

$$\dot{x} = Ax \quad , \quad x(0) = x_0 = (1,0)^T$$

note that

$$c = \Psi(0)^{-1}x_0 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

so that

$$x(t) = \Psi(t)c = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \\ e^{3t} - e^{-t} \end{pmatrix}$$

As a final note, it is possible that one of the eigenvalues is zero in this case. Suppose $\lambda_1 = 0$. Then, one solution is a constant (nontrivial) solution

$$x_1(t) = \zeta_1$$

where ζ_1 is the eigenvector associated with the zero eigenvalue. In this case, the eigenspace $E_{\lambda_1}(A) = E_0(A) = N(A)$! In other words, A is not invertible since $det(A - 0I) = 0$. These constant solutions correspond to the fixed points of $\dot{x} = Ax$ which occur on the line spanned by ζ_1.

2.2 Complex Conjugate Eigenvalues

Suppose that $A \in \mathbb{R}^{2\times 2}$ has complex eigenvalues. Such eigenvalues (being roots of a quadratic) must occur in complex conjugate pairs. Specifically, suppose that one eigenvalue λ is

$$\lambda = a + ib \quad .$$

Then there is a complex eigenvector $\zeta \in \mathbb{C}^2$ such that

$$A\zeta = \lambda\zeta$$

The complex conjugate of any complex number $z = a + ib$ is defined as:

$$\bar{z} = a - ib$$

It can easily be verified that for any two complex numbers z_1 and z_2,

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2 \quad .$$

For a vector $\zeta = (\zeta_1, \zeta_2)^T$, the conjugate $\bar{\zeta}$ is defined as:

$$\bar{\zeta} = \begin{pmatrix} \bar{\zeta}_1 \\ \bar{\zeta}_2 \end{pmatrix} \quad .$$

A similar defintion holds for matrices A but in our case, A is real so that $\bar{A} = A$.

As a result, if (λ, ζ) is an eigenvalue-eigenvector pair for A, then the calculations

$$\begin{aligned} \overline{A\zeta} &= \overline{\lambda\zeta} \\ \bar{A}\bar{\zeta} &= \bar{\lambda}\bar{\zeta} \\ A\bar{\zeta} &= \bar{\lambda}\bar{\zeta} \end{aligned}$$

show that $(\bar{\lambda}, \bar{\zeta})$ is also an eigenvalue-eigenvector pair for A.

Even though the eigenvalues and eigenvectors are complex,

$$x(t) = e^{\lambda t}\zeta$$

is still a solution of $\dot{x} = Ax$. The solution, however, is not real. To construct a Fundamental Solution Matrix, we need two linearly independent <u>real</u> solutions. Toward this end, we define the notations $Re(X)$ and $Im(X)$ to be the real and imaginary parts of X, respectively. Then, $x(t) = x_r(t) + ix_i(t)$ where $x_r = Re(x)$ and $x_i = Im(x)$. Substituting this into the differential equation one finds:

$$\dot{x_r} + i\dot{x_i} = Ax_r + iAx_i.$$

Since the real and imaginary parts of each side of this equation must match we see that real solutions can be extracted from the real and imaginary parts of $x(t)$. By writing the complex eigenvector ζ associated with $\lambda = a + ib$ as

$$\zeta = \zeta_r + i\zeta_i$$

where $\zeta_r = Re(\zeta)$ and $\zeta_i = Im(\zeta_i)$,

$$\begin{aligned} x(t) &= e^{(a+ib)t}(\zeta_r + i\zeta_i) \\ x(t) &= e^{at}(cos(bt) + isin(bt))(\zeta_r + i\zeta_i) \\ x(t) &= x_r(t) + ix_i(t) \end{aligned}$$

where

$$\begin{aligned} x_r(t) &= e^{at}\left(cos(bt)\vec{\zeta_r} - sin(bt)\vec{\zeta_i}\right) && (3) \\ x_i(t) &= e^{at}\left(sin(bt)\vec{\zeta_r} + cos(bt)\vec{\zeta_i}\right). && (4) \end{aligned}$$

Then, a Fundamental Solution Matrix can be formed by using $x_r(t)$ and $x_i(t)$ as its columns:

$$\Psi(t) = [x_r(t) \mid x_i(t)]$$

Notice that if $Re(\lambda) = a = 0$, solutions remain bounded but $x = 0$ is not attracting (neutral stability). If $Re(\lambda) < 0$,

$$x(t) = c_1x_r(t) + c_2x_i(t) \to 0 \quad , \quad as\, t \to \infty$$

demonstrating $x = 0$ is attracting (and asymptotically stable). If $Re(\lambda) > 0$, then the fixed point $\bar{x} = 0$ is unstable since the solution grows without bound.

Example 3 *Let*

$$\dot{x} = Ax = \begin{bmatrix} 1 & -1 \\ 5 & -3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic polynomial for A is

$$P(\lambda) = det(A - \lambda I) = \lambda^2 + 2\lambda + 2$$

The roots of P are the eigenvalues. In this case $\lambda = -1 + i$ is one eigenvalue (the other is $\bar{\lambda} = -1 - i$ which we don't need).

$$(A - \lambda I) = \begin{bmatrix} 2-i & -1 \\ 5 & -2-i \end{bmatrix}$$

so that $\zeta = (1, 2 - i)^T$ is a complex eigenvector associated with the eigenvalue λ. Here,

$$\zeta = \zeta_r + i\zeta_i = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + i \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

Using (3)-(4), one finds two independent (real) solutions:

$$x_r(t) = \begin{pmatrix} e^{-t} cos\, t \\ e^{-t}(2cos\, t + sin\, t) \end{pmatrix} \quad , \quad x_i(t) = \begin{pmatrix} e^{-t} sin\, t \\ e^{-t}(2sin\, t - cos\, t) \end{pmatrix}$$

A Fundamental Solution Matrix is

$$\Psi(t) = \begin{bmatrix} e^{-t} cos\, t & e^{-t} sin\, t \\ e^{-t}(2cos\, t + sin\, t) & e^{-t}(2sin\, t - cos\, t) \end{bmatrix},$$

from which $\Phi(t)$ can be computed.

2.3 Real and Equal Eigenvalues

The last case to consider is when $A \in \mathbb{R}^{2\times 2}$ has a single repeated eigenvalue. An simple example of such a matrix is:

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

whose characteristic polynomial is $P = (1 - \lambda)^2$, i.e., $\lambda = 1$ is the sole (repeated) eigenvalue.

If λ_0 is a repeated eigenvalue of $A \in \mathbb{R}^{2\times 2}$ and ζ_0 is the associated eigenvector then

$$x(t) = e^{\lambda_0 t}\zeta_0$$

is still a solution. The problem is that we do not have another eigenvalue-eigenvector pair from which to construct a second solution [3]. To find a second solution, assume that

$$y(t) = te^{\lambda_0 t}\eta^* + e^{\lambda_0 t}\eta \tag{5}$$

[3] except in the exceptional case where A is the zero matrix. Then, $\lambda_0 = 0$ and $(1,0)^T$, $(0,1)^T$ are two independent eigenvectors.

where η^* and η are vectors to be determined. Straight forward calculations reveal

$$Ay - \dot{y} = te^{\lambda_0 t}(A\eta^* - \lambda_0\eta^*) + e^{\lambda_0 t}(A\eta - \lambda_0\eta - \eta^*) \ .$$

Thus, if we choose η^* and η so that

$$(A - \lambda_0 I)\eta^* = 0 \tag{6}$$

$$(A - \lambda_0 I)\eta = \eta^* \tag{7}$$

then $y(t)$ solves $\dot{y} = Ay$. Since λ_0 is an eigenvalue of A then (6) will be satisfied by the choice $\eta^* = \zeta_0$, the eigenvector. In summary, the second solution $y(t)$ is

$$y(t) = te^{\lambda_0 t}\zeta_0 + e^{\lambda_0 t}\eta \tag{8}$$

where η is a solution of

$$(A - \lambda_0 I)\eta = \zeta_0 \ . \tag{9}$$

Then the Fundamental Solution Matrix is formed in the usual way:

$$\Psi(t) = [\, x(t) \,|y(t)\,] = [\, e^{\lambda_0 t}\zeta_0 \mid te^{\lambda_0 t}\zeta_0 + e^{\lambda_0 t}\eta\,].$$

One key issue constructing $\Psi(t)$ in such a way is the solvability of (9). In particular, one cannot simply write $\eta = (A - \lambda_0 I)^{-1}\zeta_0$ since λ_0 was chosen so that the inverse of $(A - \lambda_0 I)$ did not exist! Nevertheless, solution η of (9) may still exist [4]. Below we illustrate the procedure in an example.

Example 4 *Let*

$$\dot{x} = Ax = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

The characteristic polynomial for A is

$$P(\lambda) = det(A - \lambda I) = (\lambda - 2)^2$$

Thus $\lambda = \lambda_0 = 2$ is a repeated eigenvalue. Since

$$(A - 2I) = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix},$$

$\zeta_0 = (1, -1)^T$ is an eigenvector associated with the eigenvalue λ_0. Thus,

$$x(t) = e^{2t}\zeta_0 = \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix}$$

is a solution. To find $y(t)$ in (8) we need to find a solution η of $(A - 2I)\eta = \zeta_0$. If $\eta = (\eta_1, \eta_2)^T$, this is the same as finding a solution of:

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

[4] They won't be unique since one can can always add an element of $N(A - \lambda_0 I)$ to η and that will still be a solution.

As an augmented matrix this system is:

$$[A-2I|\zeta_0] = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

which after row reduction yields:

$$\begin{bmatrix} -1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

As a scalar equation this is equivalent to:

$$-\eta_1 - \eta_2 = 1$$

so that if $\eta_1 = -1$, we must have $\eta_2 = 0$ or

$$\eta = (-1, 0)^T .$$

Then, $y(t)$ is known:

$$y(t) = \begin{pmatrix} (t-1)e^{2t} \\ -te^{2t} \end{pmatrix}$$

Then, a Fundamental Matrix Solution is

$$\Psi(t) = [\, x(t) \; | y(t) \,] = \begin{bmatrix} e^{2t} & (t-1)e^{2t} \\ -e^{2t} & -te^{2t} \end{bmatrix}$$

Notice how the growth of $y(t)$ is faster than the growth of $x(t)$ since the exponential is multiplied by t.

2.4 Basic Linear Subspaces for Fixed Points

For the planar system

$$\dot{x} = Ax \quad , \quad A \in \mathbb{R}^{2\times 2}$$

the solution $x(t)$ and fixed point stability properties can all be determined from the eigenvalues and eigenvectors of A. If

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

the characteristic polynomial

$$P(\lambda) = det(A - \lambda I) = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$$

Written another way,

$$P(\lambda) = \lambda^2 - Tr(A)\lambda + det(A)$$

where $Tr(A) = a_{11} + a_{22}$ is the trace of the matrix A. Thus, the stability of the fixed point $\bar{x}$ is determined entirely by the two quantities $Tr(A)$ and $det(A)$. Roots of $P(\lambda)$ are:

$$\lambda_{\pm} = \frac{1}{2}\left(Tr(A) \pm \sqrt{(Tr(A))^2 - 4det(A)}\right)$$

By considering all the possible permutations of signs of $Tr(A)$ and $(Tr(A))^2 - det(A)$ one can easily deduce the following table for the stability of $\bar{x} = 0$.

Associated with $\bar{x} = 0$ we also define three linear manifolds:

Definition 1 *For $A \in \mathbb{R}^{2\times 2}$, let*

$$A\xi_k = \lambda_k \xi_k \quad , \quad \xi_k = x_k + iy_k \quad , \quad k = 1, 2$$

where x_k and y_k are the real and imaginary parts of the eigenvectors ξ_k when they are complex. Then,

$$\begin{aligned} E^s(0) &\equiv span\{x_k, y_k : Re(\lambda_k) < 0\} \\ E^c(0) &\equiv span\{x_k, y_k : Re(\lambda_k) = 0\} \\ E^u(0) &\equiv span\{x_k, y_k : Re(\lambda_k) > 0\} \end{aligned}$$

Here $E^s(0), E^c(0)$ and $E^u(0)$ are the linear stable, center and unstable manifolds associated with $\bar{x} = 0$.

$det(A) < 0$	$\bar{x} = 0$ is a saddle
$0 < det(A) \leq \frac{1}{4}(TrA)^2, TrA < 0$	$\bar{x} = 0$ is a stable node
$det(A) > \frac{1}{4}(TrA)^2, TrA < 0$	$\bar{x} = 0$ is a stable spiral
$det(A) > 0, TrA = 0$	$\bar{x} = 0$ is a center
$det(A) > \frac{1}{4}(TrA)^2, TrA > 0$	$\bar{x} = 0$ is an unstable spiral
$0 < det(A) \leq \frac{1}{4}(TrA)^2, TrA > 0$	$\bar{x} = 0$ is an unstable node
$det(A) = 0$	$\bar{x} \in N(A)$ are all fixed points