Defn: The flow $\phi(t, x_0)$ generated by $\dot{x} = f(x)$ is that function $\phi: \mathbb{R}\times\mathbb{R}^2 \to \mathbb{R}^2$ which uniquely solves the IVP

$$\dot{x} = f(x) \qquad x(0) = x_0$$

Therefore

(1) $\dfrac{\partial \phi}{\partial t} = f(\phi(t, x_0))$

(2) $\phi(0, x_0) = x_0$

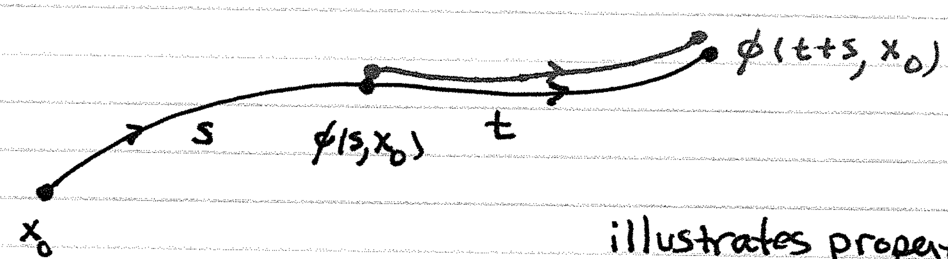
EXAMPLE Recall for $\dot{x} = Ax$ the solution is $x(t) = \Phi(t)x_0$ where $\Phi(t)$ is the fundamental matrix. Thus

$$\phi(t, x_0) = \Phi(t)x_0$$

Properties of flow functions

(a) $\phi(t+s, x_0) = \phi(t, \phi(s, x_0))$

(b) $\phi(-t, \phi(t, x_0)) = x_0$



illustrates property (a)

EXAMPLE Flow function for a nonlinear system

$$\dot{x} = -x^2 \qquad x(0) = x_0$$
$$\dot{y} = x + y \qquad y(0) = y_0$$

By direct solution techniques

$$x(t) = \frac{x_0}{1 + x_0 t} = \phi_1(t, x_0, y_0)$$

$$y(t) = y_0 e^t + e^t \int_0^t \frac{x_0}{1 + x_0 s} e^{-s}\, ds = \phi_2(t, x_0, y_0)$$

are the components of $\phi(t, x, y)$:

$$\phi(t, x_0, y_0) = \begin{pmatrix} \phi_1(t, x_0, y_0) \\ \phi_2(t, x_0, y_0) \end{pmatrix}$$

and $\phi : \underset{\substack{\uparrow \\ t}}{\mathbb{R}} \times \underset{\substack{\uparrow \\ (x_0, y_0)}}{\mathbb{R}^2} \to \mathbb{R}^2$

Defn: A homeomorphism H is a continuous invertible map $H: X \to Y$. when continuously differentiable as well H is a diffeomorphism.

Theorem* Hartman-Grobman Theorem on $\mathbb{R}^2$

(1) $f: \mathbb{R}^2 \to \mathbb{R}^2$ cont. diff on open $E \subset \mathbb{R}^2$

(2) $\phi(t, x_0)$ flow fn for $\dot{x} = f(x)$ $\quad x(0) = x_0$

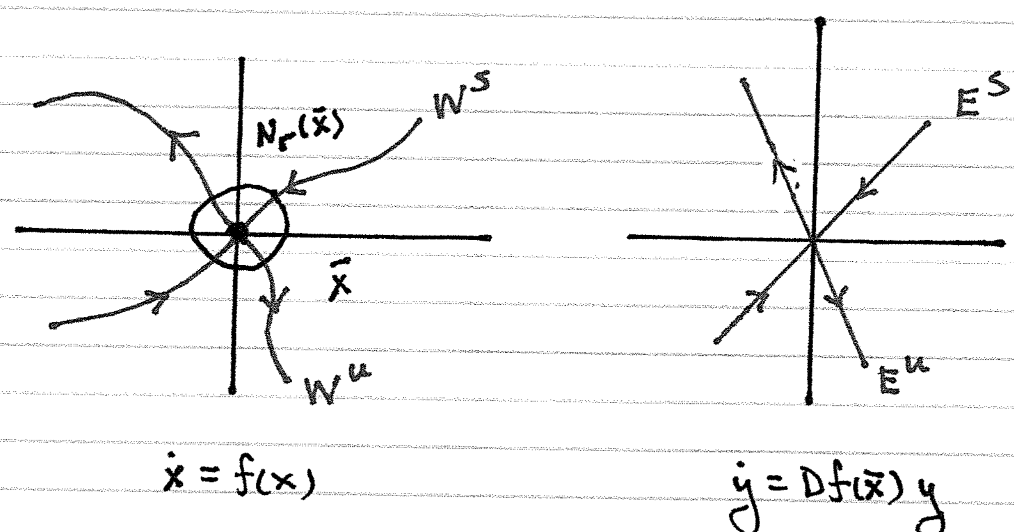
(3) $\bar{x} \in E$ hyperbolic fixed point

(4) $\psi(t, y_0)$ flow fn for $\dot{y} = Df(\bar{x})y$, $y(0) = y_0$

Then $\exists$ homeomorphism H defined on a neighbourhood $N_r(\bar{x})$ such that

$$\boxed{H(\phi(t, x_0)) = \psi(t, H(x_0))} \qquad \forall t \ni \phi(t, x_0) \in N_r(\bar{x})$$

For a saddle a casual picture looks like



* preserves direction of flow.

Global stable/unstable manifolds of fixed pts

Let $\bar{x}$ be a fixed point of $\dot{x} = f(x)$ with associated flow $\phi(t, x_0)$

$$W^s(\bar{x}) \equiv \{ x_0 : \phi(t, x_0) \to \bar{x} \text{ as } t \to \infty \}$$

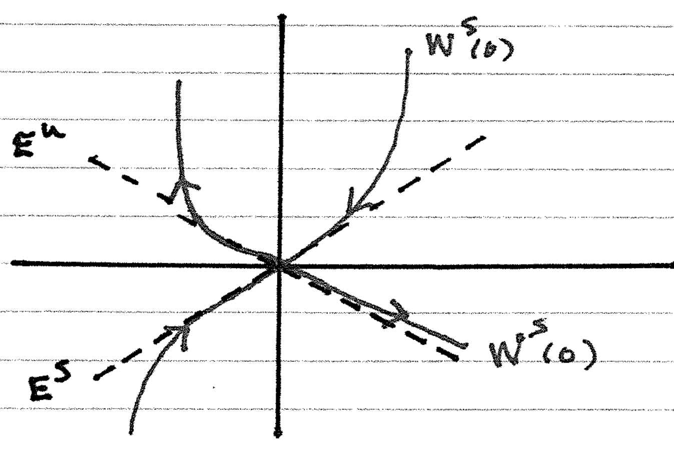
$$W^u(\bar{x}) \equiv \{ x_0 : \phi(t, x_0) \to \bar{x} \text{ as } t \to -\infty \}$$

and for our purposes the basin of attraction of a stable fixed point is

$$B(\bar{x}) \equiv W^s(\bar{x})$$

Remark: applies only to attracting fixed points.

EXAMPLE Saddle like fixed points



$W^{(s)}(0) \cap W^u(0) = \{0\}$

Near the fixed point the stable and unstable linear manifolds are tangent to W^s, W^u, respectively.

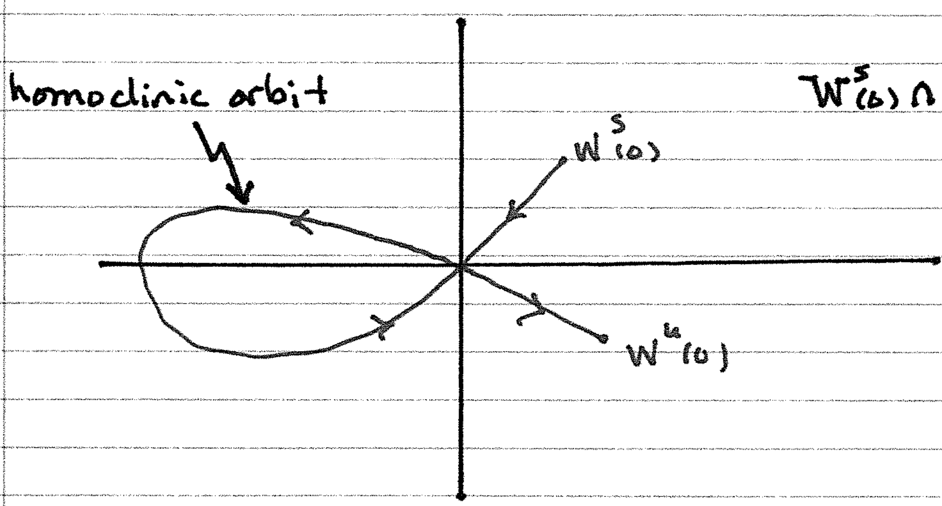
Not proven here.

EXAMPLE Homoclinic orbit

If $\bar{x}$ is a fixed point and

$$\phi(t, x_0) \to \bar{x} \quad \text{as } t \to \pm\infty$$

then the flow defines a homoclinic orbit (that is homoclinic to $\bar{x}$)



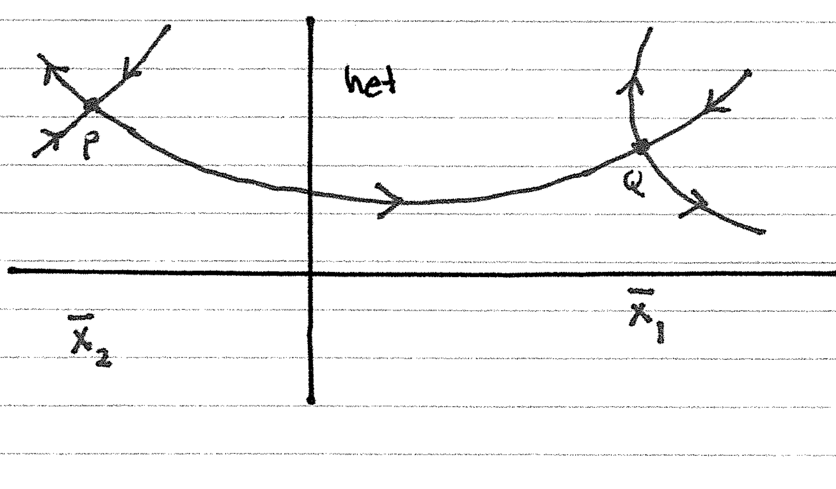
$W^s(0) \cap W^u(0)$ nontrivial

EXAMPLE Heteroclinic orbit

If $\bar{x}_1, \bar{x}_2$ are two fixed points and

$$\phi(t, x_0) \to \bar{x}_1 \quad \text{as } t \to +\infty$$
$$\phi(t, x_0) \to \bar{x}_2 \quad \text{as } t \to -\infty$$

then $\phi(t, x_0)$ defines a heteroclinic orbit



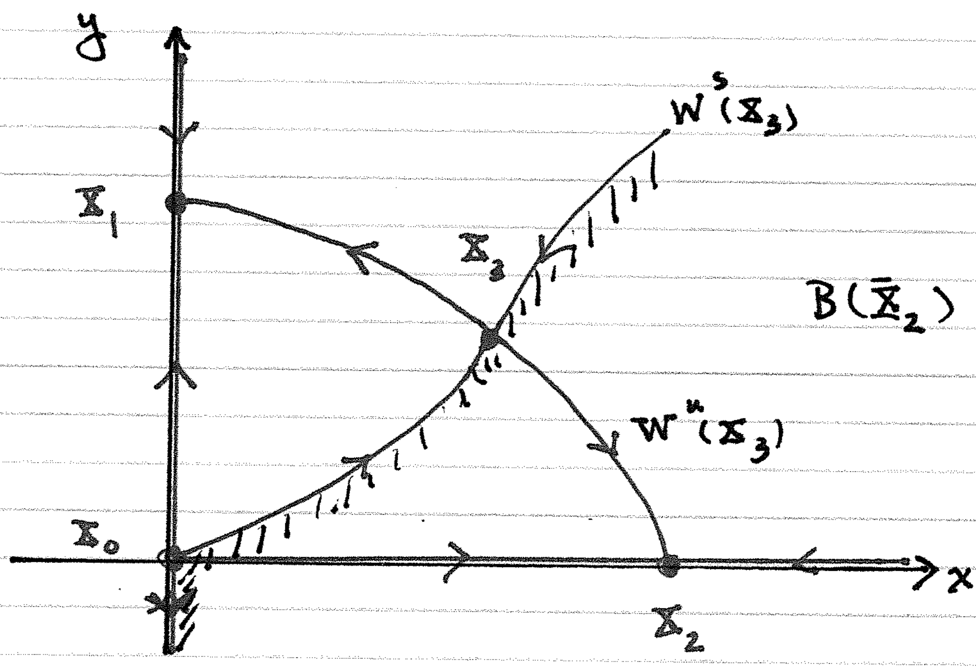
Here $W^u(\bar{x}_2) \cap W^s(\bar{x}_1)$ nontrivial.

Lotka Volterra competition model

$$\dot{x} = x(3 - x - 2y) \qquad \text{rabbits}$$

$$\dot{y} = y(2 - x - y) \qquad \text{sheep}$$

is a model of competition of different species for the same resources (see text). One can deduce the following phase portrait:



Basin of attraction of (extinction) state $\bar{x}_2$ shaded.

Linear and Phaseplane analysis for previous prob.

(i) Find fixed points:

$$x(3-x-2y) = 0$$
$$y(2-x-y) = 0$$

yields four points

$$\mathbb{X}_0 = (0,0) \quad \mathbb{X}_1 = (0,2) \quad \mathbb{X}_2 = (3,0) \quad \mathbb{X}_4 = (1,1)$$

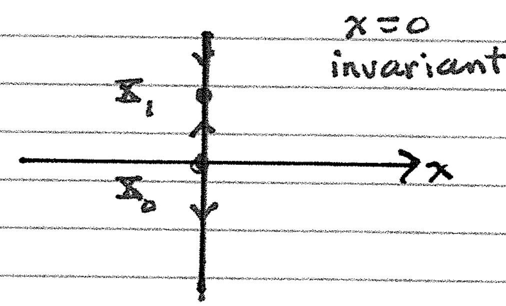
(ii) Show $x=0$ and $y=0$ are invariant sets. This means for initial conditions on the $x=0$ or y-axis the trajectory stays on $x=0$ for all time.

$$\dot{x} = x(3-x-2y)$$
$$\dot{y} = y(2-x-y)$$

On $x=0$ these read

$$\dot{x} = 0$$
$$\dot{y} = y(2-y)$$

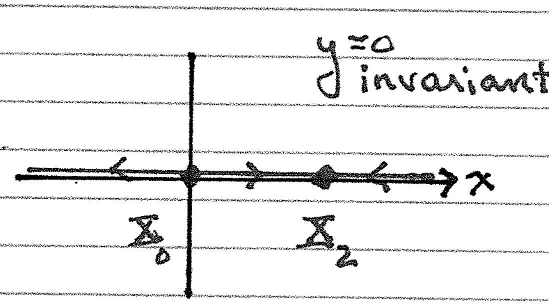
so $x(t)=0$ if $x(0)=0$.



Likewise on $y=0$ one has

$$\dot{x} = x(3-x)$$
$$\dot{y} = 0$$

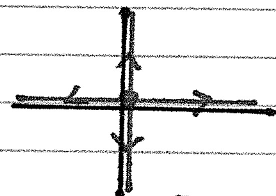
so $y(t)=0$ if $y(0)=0$



(iii) <u>Linear analysis</u>

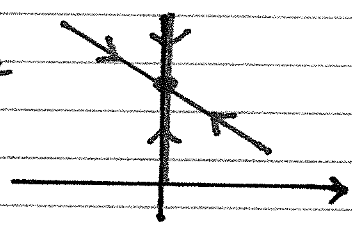
$$Df = \begin{bmatrix} 3-2x-2y & -2x \\ -y & 2-x-2y \end{bmatrix}$$

(a) $\mathbf{X}_0 = (0,0)$ $\quad Df(\mathbf{X}_0) = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$ $\quad \lambda_1 = 3 \quad \vec{\xi}_1 = (1,0)$; $\quad \lambda_2 = 2 \quad \vec{\xi}_2 = (0,1)$



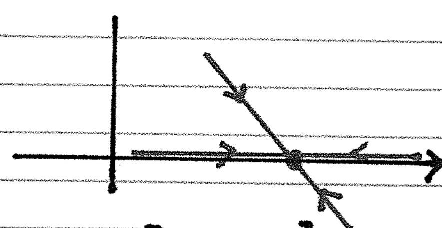
unstable node

(b) $\mathbf{X}_1 = (0,2)$ $\quad Df(\mathbf{X}_1) = \begin{bmatrix} -1 & 0 \\ -2 & -2 \end{bmatrix}$ $\quad \lambda_1 = -1 \quad \vec{\xi}_1 = (1,-2)$; $\quad \lambda_2 = -2 \quad \vec{\xi}_2 = (0,1)$



stable node

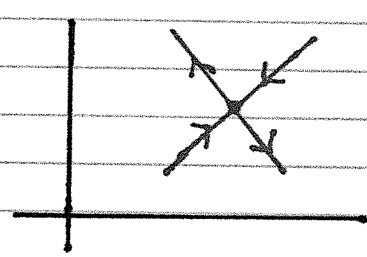
(c) $\mathbf{X}_2 = (3,0)$ $\quad Df(\mathbf{X}_2) = \begin{bmatrix} -3 & -6 \\ 0 & -1 \end{bmatrix}$ $\quad \lambda_1 = -1 \quad \vec{\xi}_1 = (-3,1)$; $\quad \lambda_2 = -3 \quad \vec{\xi}_2 = (1,0)$



stable node

(d) $\mathbf{X}_3 = (1,1)$ $\quad Df(\mathbf{X}_3) = \begin{bmatrix} -1 & -2 \\ -1 & -1 \end{bmatrix}$ $\quad \lambda = -1 \pm \sqrt{2}$

$\vec{\xi}$ complex



saddle since $\lambda_\pm$ have diff signs

Conservative Systems

Defn: A first integral $E(x)$ of $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ is a function $E : \mathbb{R}^n \to \mathbb{R}$ s.t.

(1) $E(x(t))$ is constant on solutions $x(t)$

(2) $E(x)$ is continuous (or $C^1(\Omega)$)

(3) E is <u>non-constant</u> on every open set U

Defn: A system $\dot{x} = f(x)$ is said to be conservative if it has a first integral.

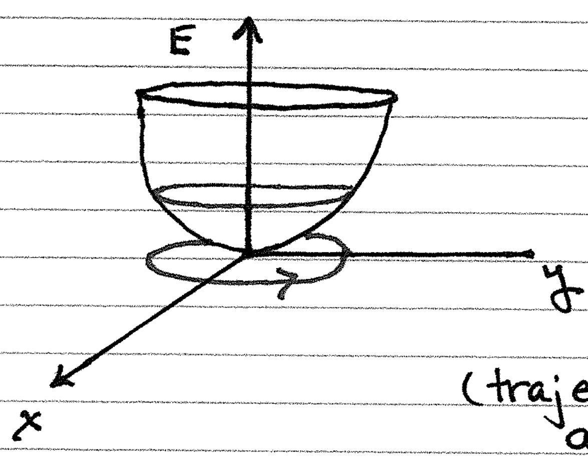
EXAMPLE Center $\boxed{\begin{array}{l} \dot{x} = -y \\ \dot{y} = x \end{array}}$

First integral is

$$E(x,y) = x^2 + y^2$$

One can verify constancy of E on solutions:

$$\dot{E} = 2x\dot{x} + 2y\dot{y} = 2x(-y) + 2y(x) = 0$$



Level sets of E are trajectories of $\dot{\underline{x}} = \underline{f}(\underline{x})$

(trajectories $E = \text{const}$ are all circles)

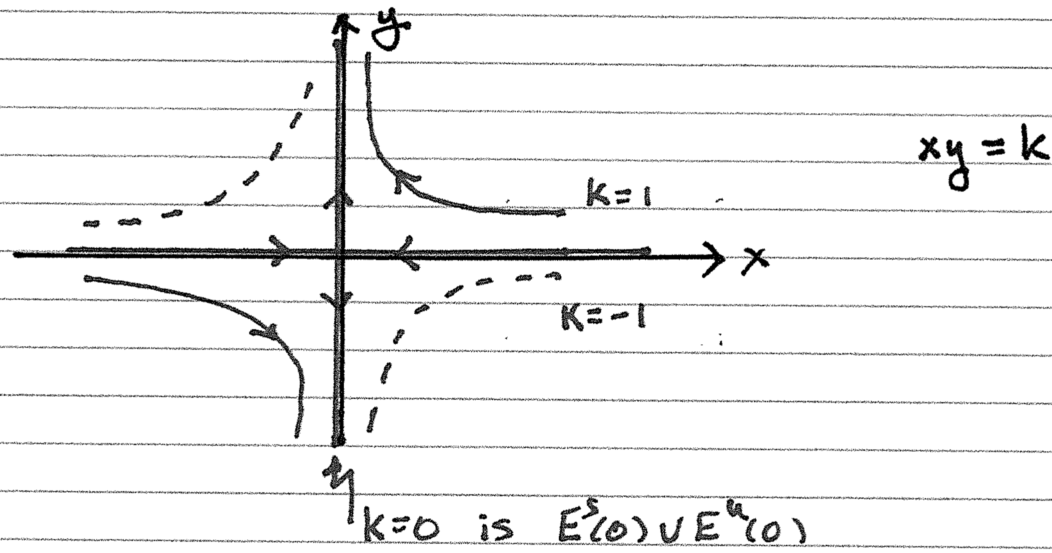
EXAMPLE Saddle

$$\dot{x} = -x$$
$$\dot{y} = y$$

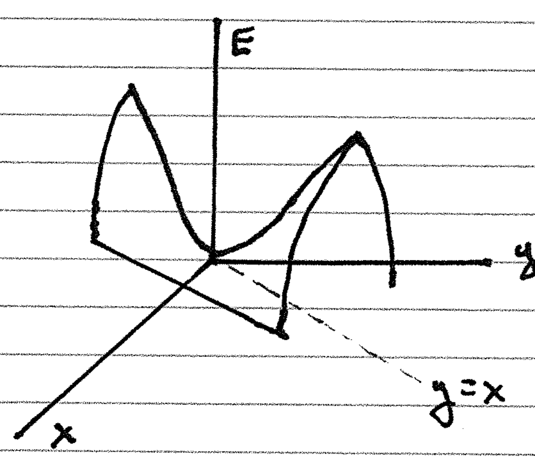
has the first integral

$$E(x,y) = xy$$

Trajectories are level sets of $E(x,y) = k$



The fixed point gets its name "saddle" from the fact the graph of $E(x,y)$ is a saddle (along $y=x$ axis)



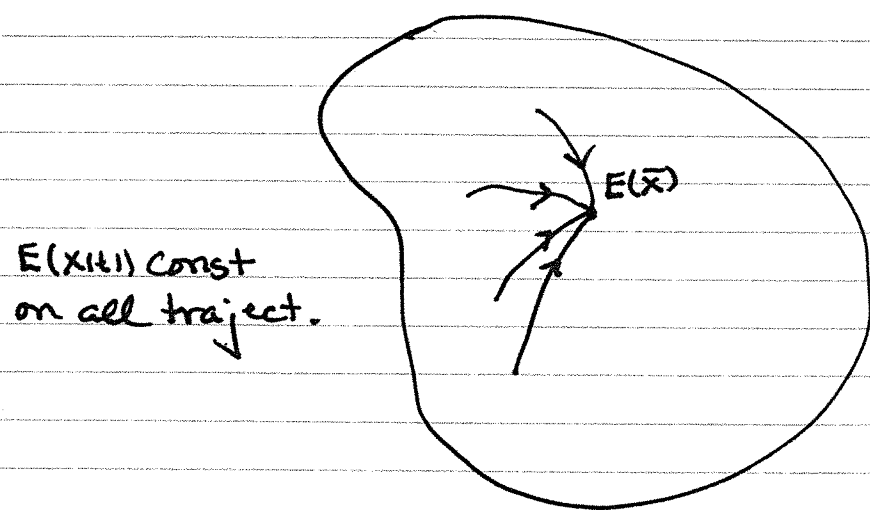
<u>Theorem</u> A conservative system $\dot{x} = f(x)$ cannot have an attracting fixed point.

<u>Proof</u> (rough idea)

Suppose to the contrary $\bar{x}$ is an attracting fixed point. Then $\exists N_\delta(\bar{x})$ such that

$$E(\phi(t, x_0)) \to E(\bar{x}) \qquad \text{as } t \to \infty$$

for all $x_0 \in N_\delta(\bar{x})$. With E continuous this means E is the same constant $E(\bar{x})$ in $N_\delta(\bar{x})$ contradicting the property that E is nonconstant on every open set.



Forces with potential functions.

Let m = mass, $F(x)$ = force acting on m. Then

$$\tfrac{1}{2} m \ddot{x} + F(x) = 0$$

wlog $m = 2$ so that

(1) $$\ddot{x} + F(x) = 0$$

Define the potential function

$$V(x) = \int^{x} F(s)\,ds$$

Multiply (1) by $\dot{x}$ and integrate in time

(2) $$\underbrace{\tfrac{1}{2}\dot{x}^2}_{\text{K.Energ.}} + \underbrace{V(x)}_{\text{P.Energy}} = E$$

From this we deduce

$$E(x,y) = \tfrac{1}{2} y^2 + V(x)$$

is a <u>first integral</u> of

$$\dot{x} = y$$
$$\dot{y} = -F(x)$$

EXAMPLE The system

$$\dot{x} = y$$
$$\dot{y} = x - x^3 = -F(x)$$

fits the previous theory. One can then derive $E(x,y)$ (not unique)

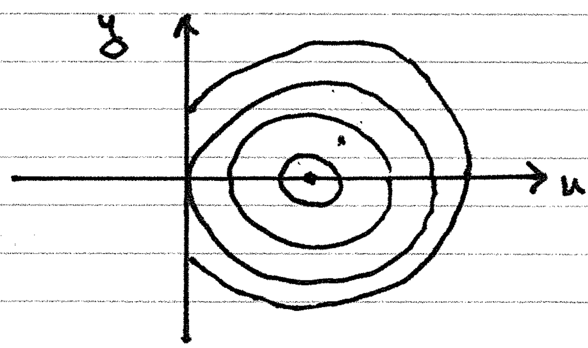
$$E(x,y) = y^2 + \tfrac{1}{2}x^4 - x^2$$

Trajectories of this conservative system are level sets of $E(x,y) = \alpha$

$$y^2 + \tfrac{1}{2}(x^2-1)^2 = \alpha - \tfrac{1}{2} \equiv \beta$$

Let $x^2 = u$ then

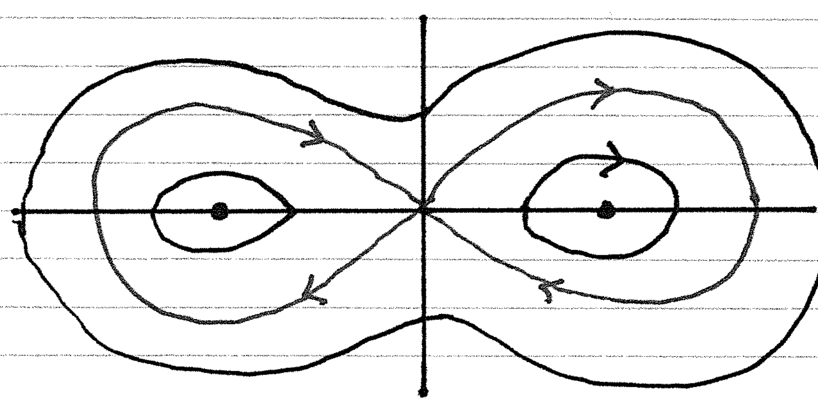
$$y^2 + \tfrac{1}{2}(u-1)^2 = \beta \qquad \text{ellipses in } (u,y)\text{-plane}$$



$u \geq 0$

Concentric ellipses

From which we deduce



"double well potential"

Planar Hamiltonian Systems

Defn: A planar system $\dot{x} = f(x)$ is Hamiltonian if $\exists\, H : \mathbb{R}^2 \to \mathbb{R}$ such that

$$(1) \qquad \dot{x}_1 = f_1(x) = -\frac{\partial H}{\partial x_2}$$

$$(2) \qquad \dot{x}_2 = f_2(x) = \frac{\partial H}{\partial x_1}$$

$H = H(x_1, x_2)$ is called the Hamiltonian.

Remark: If H is a Hamiltonian then it is conservative with $E = H$ as the first integral since

$$\frac{dH}{dt} = \frac{\partial H}{\partial x_1}\dot{x}_1 + \frac{\partial H}{\partial x_2}\dot{x}_2 = 0 \qquad \text{by (1)-(2)}$$

Theorem (Calc 3) If $F : \mathbb{R}^2 \to \mathbb{R}^2$ is C^1 on a simply connected domain

$$\vec{F} = \vec{\nabla}\phi \iff \vec{\nabla} \times \vec{F} = \vec{0}$$

One applies this theorem to $\vec{F} = (f_2, -f_1)$ and $\phi = H$ we have

$$\dot{x} = f(x) \text{ Hamiltonian} \iff \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 0$$

$$(3) \qquad \dot{x} = f(x) \text{ Hamiltonian} \iff \vec{\nabla} \cdot \vec{f} = 0$$

Statement (3) yields a simple test to see if (1)-(2) is Hamiltonian given f.

EXAMPLE Show

$$(1) \qquad \dot{x} = 2y - \tfrac{1}{2}x^2$$
$$(2) \qquad \dot{y} = xy + 3x^2$$

is Hamiltonian and find $H(x,y)$. First take divergence of vector field

$$\vec{\nabla}\cdot f = \frac{\partial}{\partial x}(2y - \tfrac{1}{2}x^2) + \frac{\partial}{\partial y}(xy + 3x^2) = 0 \quad \checkmark$$

hence (1)-(2) is Hamiltonian. To find $H(x,y)$ we solve

$$(3) \qquad -\frac{\partial H}{\partial y} = 2y - \tfrac{1}{2}x^2$$
$$(4) \qquad \frac{\partial H}{\partial x} = xy + 3x^2$$

Integrate (3) in y to get

$$(5) \qquad H = -y^2 + \tfrac{1}{2}yx^2 + \phi(x)$$

for some as yet to be found $\phi(x)$. Use (5) in (4)

$$\cancel{xy} + \phi'(x) = \cancel{xy} + 3x^2$$
$$\phi'(x) = 3x^2$$
$$(6) \qquad \phi(x) = x^3$$

Use (6) in (5) to conclude

$$H(x,y) = x^3 + \tfrac{1}{2}x^2y - y^2 \qquad \square$$

EXAMPLE Not all conservative systems are Hamiltonian

$$(1) \qquad \dot{x} = y$$

$$(2) \qquad \dot{y} = -x^2 y$$

One can readily verify $E = \frac{1}{3}x^3 + y$ is a first integral

$$\dot{E} = x^2\dot{x} + \dot{y} = yx^2 - yx^2 = 0 \qquad \checkmark$$

but (1)-(2) is not Hamiltonian since $\vec{\nabla}\cdot f \neq 0$, i.e.

$$\frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(-x^2 y) \neq 0$$

Summary of system types