Nonlinear Centers \( \dot{x} = f(x) \)

when a fixed point \( \bar{x} \) is hyperbolic the eigenvalues of the Jacobian determine what the flow looks like near \( \bar{x} \). But, if \( \bar{x} \) is not hyperbolic this is NOT the case. In particular, if \( \bar{x} \) is a linear center where the eigenvalues of \( Df(\bar{x}) \) are purely imaginary, one does NOT necessarily know if \( \bar{x} \) is a nonlinear center.

We need an extra tool for such cases.

**Theorem** let \( \dot{x} = f(x), \ x \in \mathbb{R}^2 \) and

1. \( E(x) \) is a first integral
2. \( \bar{x} \) an isolated fixed point
3. \( \bar{x} \) a local min (max) of \( E \)
4. \( E(x) \in C^2(\mathbb{R}^2) \)

Then, near \( \bar{x} \), all trajectories are closed.

\[ \text{Pf.} \quad \text{Near } \bar{x}: \]

![Diagram](image.png)

level curve of \( E \)

trajectory of \( \dot{x} = f(x) \)
Question: If we know the first integral $E(x)$ and $\bar{x}$ is a fixed point of $\dot{x} = f(x)$ then does $\nabla E = 0$?

This is relevant since for $\bar{x}$ to be a local min of $E$ one must have $\nabla E(\bar{x}) = 0$.

The answer is no, not generally:

$$\dot{x} = y$$
$$\dot{y} = -x^2 y$$

has a line of equilibria $(\bar{x}, 0)$, $\bar{x} \in \mathbb{R}$ and is conservative with

$$E = \frac{1}{3} x^3 + y$$

where $\nabla E = (x^2, 1) \neq (0, 0)$ for any $(x, y)$.

Second Derivative Test

Suppose $\nabla E(\bar{x}) = 0$ and define discriminant at $\bar{x}$

$$D \equiv \frac{\partial^2 E}{\partial x_1^2} \frac{\partial^2 E}{\partial x_2^2} - \left( \frac{\partial^2 E}{\partial x_1 \partial x_2} \right)^2$$

Then

$$D(\bar{x}) < 0 \quad \bar{x} \text{ saddle}$$
$$D(\bar{x}) > 0 \quad \frac{\partial^2 E}{\partial x_1^2} > 0 \quad \bar{x} \text{ local min}$$
$$D(\bar{x}) > 0 \quad \frac{\partial^2 E}{\partial x_2^2} > 0 \quad \bar{x} \text{ local max}$$
$$D(\bar{x}) = 0 \quad \text{test fails}$$
Nonlinear centers for Hamiltonian systems.

Recall not all conservative systems are Hamiltonian. If
\begin{align*}
(1) \quad \dot{x} &= -\frac{\partial H}{\partial y}(x,y) = f_1(x,y) \\
(2) \quad \dot{y} &= \frac{\partial H}{\partial x}(x,y) = f_2(x,y)
\end{align*}

then \( E = H \) is a first integral and for any fixed point of (1)-(2)

\[(\bar{x}, \bar{y}) \text{ fixed point } \iff \nabla H(\bar{x}, \bar{y}) = 0\]

To apply the nonlinear center theorem we compute
\[
Df = \begin{bmatrix}
\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\
\frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}
\end{bmatrix} = \begin{bmatrix}
-\frac{\partial^2 H}{\partial x \partial y} & -\frac{\partial^2 H}{\partial y^2} \\
\frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial y}
\end{bmatrix}
\]

Note \( \text{Tr}(Df) = 0 \) so linear stability of fixed points is determined solely by
\[
\det(Df) = D(\bar{x}, \bar{y}) = (H_{xx} H_{yy} - H_{xy}^2)
\]

which is the discriminant in the nonlinear center theorem. Either \( D < 0 \Rightarrow (\bar{x}, \bar{y}) \text{ saddle or } D > 0 \Rightarrow (\bar{x}, \bar{y}) \text{ a center.} \)

fixed points of planar Hamiltonian are either saddles or nonlinear centers.
**Example**

\[
\begin{align*}
  x_1 &= x_2 \\
  x_2 &= x_1 - x_1^2 
\end{align*}
\]

Has two fixed points \( X_0 = (0,0) \) and \( X_1 = (1,0) \). Can compute the Jacobian

\[
Df(X_0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}
\]

Since \( \det Df(X_0) < 0 \), \( X_0 \) is a (hyperbolic) saddle by linear stability analysis. The other is a linear center, which is not hyperbolic.

To show the linear center is indeed a center we note (a) it is not a saddle and (b)

\[
\nabla \cdot f = \frac{\partial}{\partial x_1} (x_2) + \frac{\partial}{\partial x_2} (x_1 - x_1^2) = 0
\]

⇒ system Hamiltonian hence \( X_1 \) is a center.

From

\[
\begin{align*}
  -\frac{\partial H}{\partial x_2} &= x_2 \\
  \frac{\partial H}{\partial x_1} &= x_1 - x_1^2
\end{align*}
\]

we can additionally find the Hamiltonian:

\[
H = -\frac{1}{2} x_2^2 + \frac{1}{2} x_1^2 - \frac{1}{3} x_1^3
\]

\[\xrightarrow{\text{level sets } H.}\]
Reversible Systems

**Defn:** A planar system

(1) \[ \dot{x} = f(x, y) \]
(2) \[ \dot{y} = g(x, y) \]

is reversible if it is invariant under the transformation \( t \to -t, \ y \to -y \)

**Remark:** If one lets \( z = -t, \ z' = -y \) then (1)-(2) becomes (for \( \gamma' = \frac{d\gamma}{dt} \))

(3) \[ x' = -f(x, z) \]
(4) \[ z' = g(x, z) \]

To be invariant under the transformation \( f(x, y) \) must be odd in \( y \) but \( g(x, y) \) must be even in \( y \) so that (3)-(4) becomes

\[ x' = f(x, z) \]
\[ z' = g(x, z) \]

**EXAMPLE** Reversible

\[ \dot{x} = \sin y - y^3 \quad \text{odd in } y \]
\[ \dot{y} = xy^2 \quad \text{even in } y \]

**EXAMPLE** Not reversible

\[ \dot{x} = x + y \]
\[ \dot{y} = xy^3 \]
Property of reversible systems

If one knows a trajectory in the upper \((y > 0)\) plane, the system must have the same (reflected) trajectory in the lower half \((y \leq 0)\) plane, but traversed in the opposite direction.

where the reflection operation is

\[ R(x, y) = (x, -y) \]

Two key uses of reversibility are

(a) can show linear centers are (true) nonlinear centers

(b) can prove existence of homoclinic \(X\) orbits and heteroclinic orbits.
Theorem  Let $\dot{x} = f(x)$ where $x \in \mathbb{R}^2$ and $f(x)$ is smooth. If 

(i) $\bar{x} = (\bar{x}_1, 0)$ isolated fixed point 

(ii) $\bar{x}$ a linear center 

(iii) $\dot{x} = f(x)$ reversible 

then all trajectories sufficiently close to $\bar{x}$ are closed (nonlinear center).

Pf (very rough sketch) (Strogatz) 

One can show that near $\bar{x}$ there is a trajectory like 

From the reversibility property we conclude
EXAMPLE Show the system
\[
\begin{align*}
\dot{x} &= y - y^3 \\
\dot{y} &= -x - y^2
\end{align*}
\] (odd in $y$) (even in $y$)

has a nonlinear center. Note $\nabla \cdot f = -2y \neq 0$
hence is not Hamiltonian. Fixed points

\[
X_0 = (0,0) \quad \text{linear center} \quad \{ \text{calculations omitted} \}
\]

\[
X_{\pm} = (-1, \pm 1) \quad \text{saddles}
\]

Since $X_0$ is on the $x$-axis and the system
is reversible the previous theorem
implies $X_0$ is a true (nonlinear) center.

Phase portrait consistent with all info:

- Nonlinear centers
- Heteroclinic orbit from $X_+$ to $X_-$
**Example** Use reversibility to show

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= x - x^2
\end{align*}
\]

has a homoclinic orbit in right plane \(x > 0\).

One can show the fixed points (and linear stability) are

\[
\begin{align*}
X_0 &= (0, 0) & \text{saddle} \\
X_1 &= (1, 0) & \text{linear center}
\end{align*}
\]

Consider the nullclines

\[
\begin{align*}
\frac{\partial y}{\partial x} &= \frac{\dot{y}}{\dot{x}} = \frac{x - x^2}{y} \quad \text{Arrows show direction of flow.} \\
\frac{\partial y}{\partial y} &= 0 \\
1 &= \frac{\dot{x}}{\dot{y}} = \frac{x - x^2}{x} \\
\end{align*}
\]

From the above we conclude \(W_u\) must eventually cross \(x\)-axis.

\[
\begin{align*}
\text{this by reversibility.}
\end{align*}
\]