Nonlinear Centers $\dot{x} = f(x)$

when a fixed point $\bar{x}$ is hyperbolic the eigenvalues of the Jacobian determine what the flow looks like near $\bar{x}$. But, if $\bar{x}$ is not hyperbolic this is NOT the case. In particular, if $\bar{x}$ is a <u>linear center</u> where the eigenvalues of $Df(\bar{x})$ are purely imaginary, one does NOT necessary know if $\bar{x}$ is a <u>nonlinear center</u>

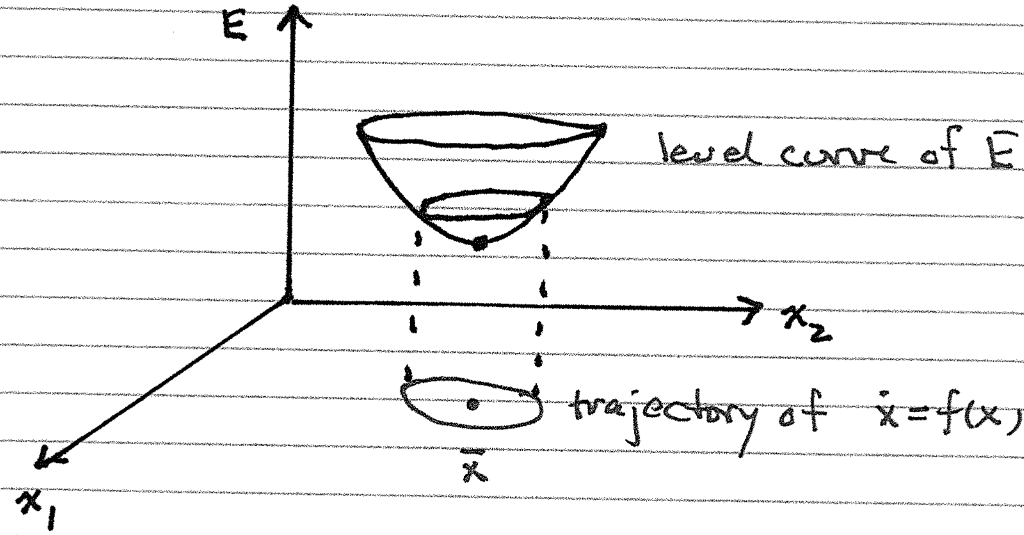
We need an extra tool for such cases.

<u>Theorem</u> Let $\dot{x} = f(x)$, $x \in \mathbb{R}^2$ and

(1) $E(x)$ is a first integral
(2) $\bar{x}$ an isolated fixed point
(3) $\bar{x}$ a local min (max) of E
(4) $E(x) \in C^2(\mathbb{R}^2)$

Then, near $\bar{x}$, all trajectories are closed

Pf/ Near $\bar{x}$:



Question: If we know the first integral $E(x)$ and $\bar{x}$ is a fixed point of $\dot{x} = f(x)$ then does $\nabla E = 0$?

This is relevant since for $\bar{x}$ to be a local min of E one must have $\nabla E(\bar{x}) = 0$.

The answer is no, not generally:

$$\dot{x} = y$$
$$\dot{y} = -x^2 y$$

has a line of equilibria $(\bar{x}, 0)$, $\bar{x} \in \mathbb{R}$ and is conservative with

$$E = \tfrac{1}{3}x^3 + y$$

where $\nabla E = (x^2, 1) \neq (0,0)$ for any (x,y).

Second Derivative Test

Suppose $\nabla E(\bar{x}) = 0$ and define discriminant at $\bar{x}$

$$D \equiv \frac{\partial^2 E}{\partial x_1^2} \frac{\partial^2 E}{\partial x_2^2} - \left(\frac{\partial^2 E}{\partial x_1 \partial x_2}\right)^2$$

Then

$D(\bar{x}) < 0$		$\bar{x}$ saddle
$D(\bar{x}) > 0$	$\frac{\partial^2 E}{\partial x_1^2} > 0$	$\bar{x}$ local min
$D(\bar{x}) > 0$	$\frac{\partial^2 E}{\partial x_2^2} > 0$	$\bar{x}$ local max
$D(\bar{x}) = 0$		test fails

Nonlinear centers for Hamiltonian systems

Recall not all conservative systems are Hamiltonian. If

$$(1) \qquad \dot{x} = -\frac{\partial H}{\partial y}(x,y) = f_1(x,y)$$

$$(2) \qquad \dot{y} = \frac{\partial H}{\partial x}(x,y) = f_2(x,y)$$

then $E = H$ is a first integral and for any fixed point of (1)-(2)

$$\boxed{(\bar{x},\bar{y}) \text{ fixed point } \Leftrightarrow \nabla H(\bar{x},\bar{y}) = 0}$$

To apply the nonlinear center theorem we compute

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} -\frac{\partial^2 H}{\partial x \partial y} & -\frac{\partial^2 H}{\partial y^2} \\ \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial y} \end{bmatrix}$$

Note $\mathrm{Tr}(Df) = 0$ so linear stability of fixed points is determined solely by

$$\det(Df) = D(\bar{x},\bar{y}) = (H_{xx} H_{yy} - H_{xy}^2)$$

which is the discriminant in the nonlinear center theorem. Either $D < 0 \Rightarrow (\bar{x},\bar{y})$ saddle or $D > 0 \Rightarrow (\bar{x},\bar{y})$ a center.

> fixed points of planar Hamiltonian are either saddles or nonlinear centers.

EXAMPLE

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_1 - x_1^2$$

Has two fixed points $\mathbb{X}_0 = (0,0)$ and $\mathbb{X}_1 = (1,0)$
Can computer the Jacobian

$$Df(\mathbb{X}_0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Since $\det Df(\mathbb{X}_0) < 0$, $\mathbb{X}_0$ is a (hyperbolic) saddle by linear stability analysis. The other is a linear center, which is not hyperbolic. To show the linear center is indeed a center we not (a) it is not a saddle and (b)

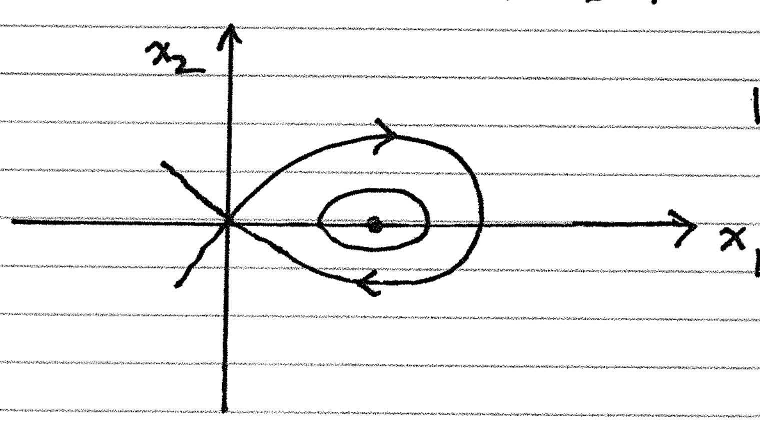
$$\nabla \cdot f = \frac{\partial}{\partial x_1}(x_2) + \frac{\partial}{\partial x_2}(x_1 - x_1^2) = 0$$

$\Rightarrow$ system Hamiltonian hence $\underline{\mathbb{X}_1}$ $\underline{\text{is}}$ a center!
From

$$-\frac{\partial H}{\partial x_2} = x_2$$
$$\frac{\partial H}{\partial x_1} = x_1 - x_1^2$$

we can additionally find the Hamiltonian:

$$H = -\tfrac{1}{2}x_2^2 + \tfrac{1}{2}x_1^2 - \tfrac{1}{3}x_1^3$$



level sets H.

Reversible Systems

Defn: A planar system

$$(1) \qquad \dot{x} = f(x,y)$$
$$(2) \qquad \dot{y} = g(x,y)$$

is reversible if it is invariant under the transformation $t \to -t$, $y \to -y$

Remark: If one lets $\tau = -t$, $z = -y$ then (1)-(2) becomes (for $(\ )' = \frac{d}{d\tau}(\ )$)

$$(3) \qquad x' = -f(x,-z)$$
$$(4) \qquad z' = g(x,-z)$$

To be invariant under the transformation $f(x,y)$ must be odd in y but $g(x,y)$ must be even in y so that (3)-(4) becomes

$$x' = f(x,z)$$
$$z' = g(x,z)$$

EXAMPLE Reversible

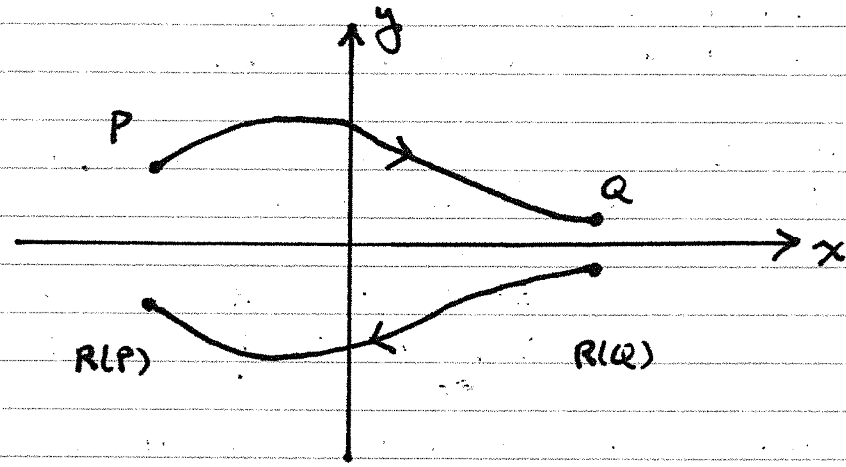
$$\dot{x} = \sin y - y^3 \qquad \text{odd in } y$$
$$\dot{y} = xy^2 \qquad \text{even in } y$$

EXAMPLE Not reversible

$$\dot{x} = x + y$$
$$\dot{y} = xy^3$$

Property of reversible systems

If one knows a trajectory in the upper ($y \geq 0$) plane the system must have the same (reflected) trajectory in the lower half ($y \leq 0$) plane but traversed in the opposite direction.



where the reflection operation is

$$R(x,y) = (x,-y)$$

Two key uses of reversibility are

(a) can show linear centers are (true) nonlinear centers

(b) can prove existence of homoclinic orbits and heteroclinic orbits.

Theorem Let $\dot{x} = f(x)$ where $x \in \mathbb{R}^2$ and $f(x)$ is smooth. If

(i) $\bar{x} = (\bar{x}_1, 0)$ isolated fixed point

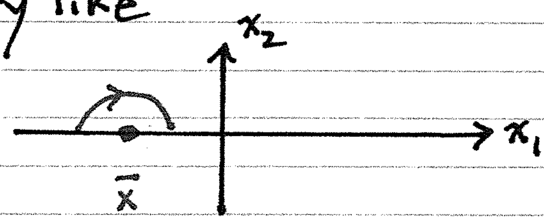
(ii) $\bar{x}$ a linear center

(iii) $\dot{x} = f(x)$ reversible

then all trajectories sufficiently close to $\bar{x}$ are closed (nonlinear center).

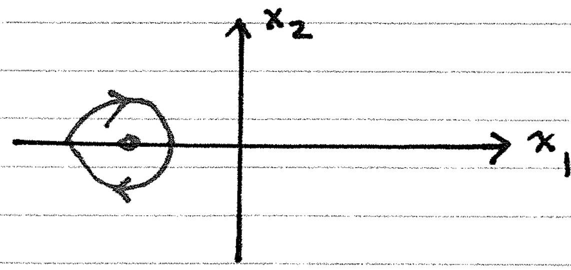
Pf (very rough sketch) (Strogatz)

One can show that near $\bar{x}$ there is a trajectory like



(lin. center)

From the reversibility property we conclude



$\square$

EXAMPLE Show the system

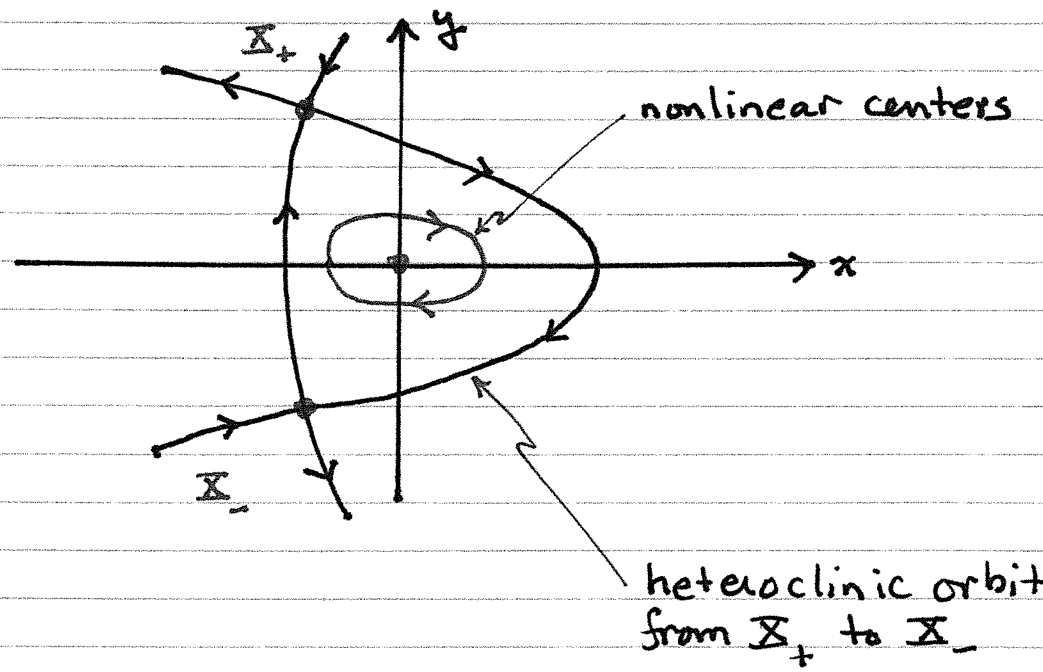
$$\dot{x} = y - y^3 \qquad \text{(odd in } y\text{)}$$
$$\dot{y} = -x - y^2 \qquad \text{(even in } y\text{)}$$

has a nonlinear center. Note $\vec{\nabla} \cdot f = -2y \neq 0$ hence is not Hamiltonian. Fixed points

$$\left.\begin{array}{ll} \mathbf{X}_0 = (0,0) & \text{linear center} \\ \mathbf{X}_\pm = (-1, \pm 1) & \text{saddles} \end{array}\right\} \text{calculations omitted}$$

Since $\mathbf{X}_0$ is on the x-axis and the system is reversible the previous theorem implies $\mathbf{X}_0$ is a true (nonlinear) center.

Phase portrait consistent with all info:



EXAMPLE Use reversibility to show

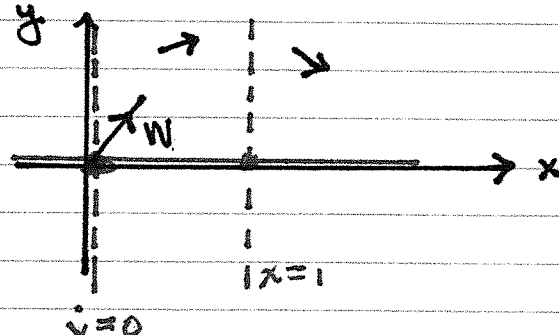
$$\dot{x} = y$$
$$\dot{y} = x - x^2$$

has a homoclinic orbit in right plane $x \geq 0$.

One can show the fixed points (and linear stability) are

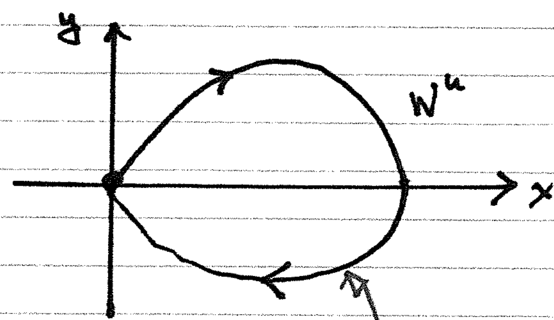
$\underline{X}_0 = (0,0)$ saddle
$\underline{X}_1 = (1,0)$ linear center

Consider the nullclines



Arrows show direction of flow. W_s near origin indicated.

From the above we conclude W must eventually cross x-axis



this by reversibility.