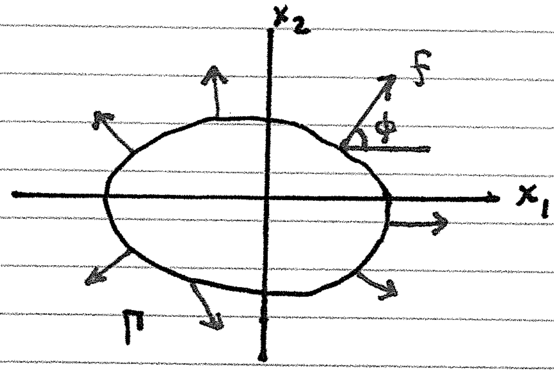
Index $\qquad \dot{x} = f(x) \qquad x \in \mathbb{R}^2$

Let Γ be a (piecewise) smooth simple closed curve.



The index $I(\Gamma)$ of f about Γ is the total (average) net change of ϕ as one traverses Γ in a counterclockwise direction. We write

(1) $$I(\Gamma) = \frac{1}{2\pi}[\phi]_\Gamma$$

Here $[\phi]_\Gamma$ is the net change in the angle ϕ. Thus we see

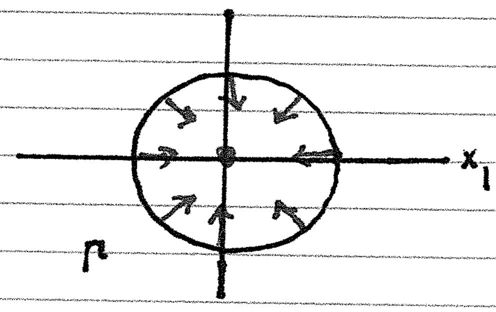
(2) $I(\Gamma)$ is an integer

representing the net number of times f rotates counterclockwise along Γ.

Remark

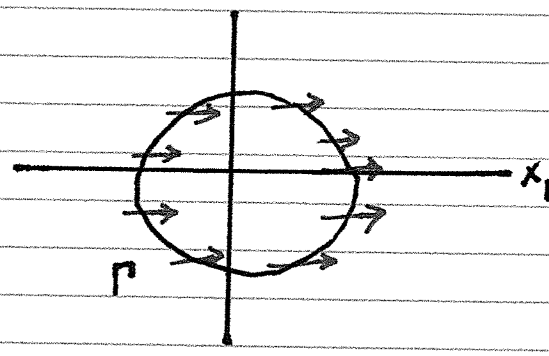
$$\phi = \tan^{-1}\left(\frac{f_2}{f_1}\right)$$

EXAMPLE Stable node



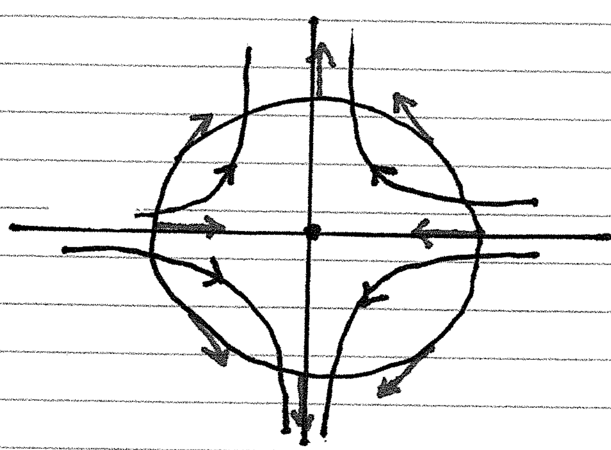
$I(\Gamma) = +1$

EXAMPLE Constant vector field



$I(\Gamma) = 0$

EXAMPLE Saddle



$I(\Gamma) = -1$

note the rotation is clockwise (once) hence the index is negative.

EXAMPLE

$$\dot{x} = 2x^2 - 1$$
$$\dot{y} = 2xy$$

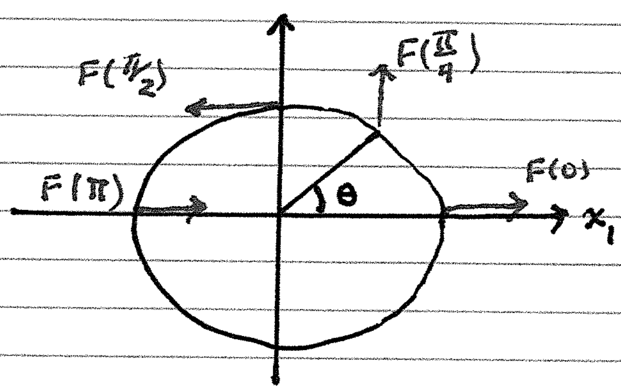
Compute the index $I(\Gamma)$ where Γ is the unit circle. Parametrize Γ:

$$\Gamma: \quad (x(\theta), y(\theta)) = (\cos\theta, \sin\theta)$$

$$f\Big|_{\Gamma} = (2\cos^2\theta - 1, \; 2\sin\theta\cos\theta)$$

Using trig identities

$$F(\theta) = f\Big|_{\Gamma} = (\cos 2\theta, \sin 2\theta)$$



contains two fixed points $\left(\pm\frac{1}{\sqrt{2}}, 0\right)$

Clearly the vector field F rotates twice counterclockwise hence

$$I(\Gamma) = +2$$

Line integral formulation of index

One can show

(1) $$I(\Gamma) = \frac{1}{2\pi} \oint_\Gamma \vec{F} \cdot d\vec{R} \qquad d\vec{R} = (d\mathbb{X}, d\mathbb{Y})$$

where

$$\mathbb{X}(s) \equiv f_1(x(s), y(s))$$

$$\mathbb{Y}(s) \equiv f_2(x(s), y(s))$$

note $\mathbb{X} = \mathbb{Y} = 0$ at fixed points

for Γ parametrized by $\vec{r}(s) = (x(s), y(s))$. And,

$$\vec{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} \frac{-\mathbb{Y}}{\mathbb{X}^2+\mathbb{Y}^2} \\ \frac{\mathbb{X}}{\mathbb{X}^2+\mathbb{Y}^2} \end{pmatrix}$$

Remark: Because the index is also a line integral, one may use Green's theorem and multivariate calculus to prove many things about $I(\Gamma)$.

An alternate way of writing (1) is

$$I(\Gamma) = \frac{1}{2\pi} \oint_\Gamma \frac{\mathbb{X}\,d\mathbb{Y} - \mathbb{Y}\,d\mathbb{X}}{\mathbb{X}^2+\mathbb{Y}^2}$$

Index properties (without proof)

① If Γ can be continuously deformed into Γ' without passing through a fixed point

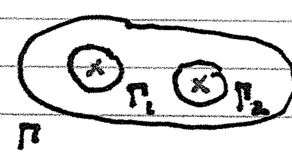
$$I(\Gamma) = I(\Gamma')$$

② If Γ encloses no fixed points

$$I(\Gamma) = 0$$

③ If Γ encloses n isolated fixed points

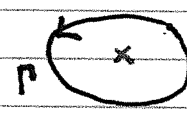
$$I(\Gamma) = \sum_{k=1}^{n} I(\Gamma_k)$$



$\times$ = fx pt

④ If $\bar{x}$ is a saddle

$$I(\Gamma) = -1$$



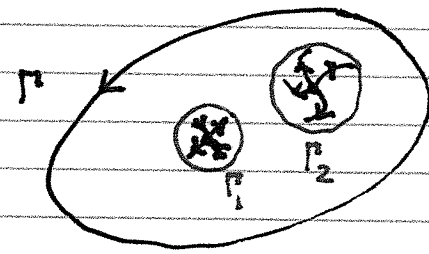
⑤ If $\bar{x}$ is not a saddle but is isolated hyperbolic or a center then

$$I(\Gamma) = +1$$

⑥ If Γ is a closed periodic orbit of $\dot{x} = f(x)$ then

$$I(\Gamma) = 1$$

EXAMPLE Suppose Γ is a closed orbit of $\dot{x} = f(x)$. Can Γ contain exactly one saddle and one node?



not possible

Must have

$$I(\Gamma) = I(\Gamma_1) + I(\Gamma_2)$$

$$I(\Gamma) \neq -1 \quad + \quad 1 \qquad = 0$$

since $I(\Gamma) = 1$ for closed orbits.

EXAMPLE Any closed orbit must contain at least one fixed point.

EXAMPLE A closed orbit contains only hyperbolic fixed points only one of which is a saddle.
How many fixed points are there?

$$I(\Gamma) = \sum_{K=1}^{n} I(\Gamma_K) + I(\Gamma_0) \quad \leftarrow \text{surrounds saddle}$$

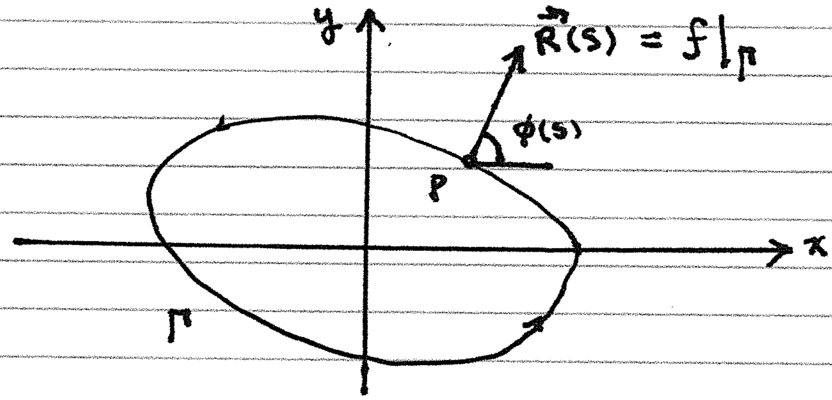
$$1 = n - 1$$

Hence $n = 2 \Rightarrow$ three fixed points.

Line integral derivation

First we define a parametrization of Γ

$$\vec{r}(s) = x(s)\hat{i} + y(s)\hat{j} \qquad s \in [0, T]$$



where $\vec{R}(s)$ is f evaluated on Γ at P

$$\vec{R}(s) = \begin{pmatrix} X(s) \\ Y(s) \end{pmatrix} = \begin{pmatrix} f_1(x(s), y(s)) \\ f_2(x(s), y(s)) \end{pmatrix}$$

At every point P on Γ

(1) $$\tan \phi(s) = \frac{Y(s)}{X(s)}$$

Differentiating (1) in s with $(\ )' = \frac{d}{ds}(\ )$

(2) $$\sec^2\phi \frac{d\phi}{ds} = \frac{XY' - YX'}{X^2}$$

but since

$$\sec^2\phi = \frac{X^2 + Y^2}{X^2}$$

equation (2) becomes

$$(3) \qquad \frac{d\phi}{ds} = \frac{X Y' - Y X'}{X^2 + Y^2}$$

Since the index is the change of ϕ over Γ divided by 2π

$$(4) \qquad I(\Gamma) = \frac{1}{2\pi}[\phi]\Big|_{\Gamma} = \frac{1}{2\pi}\int_0^T \left(\frac{d\phi}{ds}\right) ds$$

Therefore (3)-(4) $\Rightarrow$

$$I(\Gamma) = \frac{1}{2\pi}\int_0^T \frac{X(s)Y'(s) - Y(s)X'(s)}{X(s)^2 + Y(s)^2}\, ds$$

This line integral is just

$$I(\Gamma) = \frac{1}{2\pi}\int_{\Gamma} \vec{F}\cdot d\vec{R} \qquad \frac{d\vec{R}}{ds} = (X', Y')$$

where

$$\vec{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} \frac{-Y}{X^2+Y^2} \\ \frac{X}{X^2+Y^2} \end{pmatrix} \qquad \square$$

EXAMPLE Use the integral form of $I(\Gamma)$ to compute the index of the saddle

$$\dot{x} = -x$$
$$\dot{y} = y$$

Choose Γ to be the unit circle $\vec{r}(\theta) = (\cos\theta, \sin\theta)$

$$\vec{R}(\theta) = (-\cos\theta, \sin\theta) \qquad \theta \in [0, 2\pi]$$

$$\frac{d\vec{R}}{d\theta} = (\sin\theta, \cos\theta) = (X', Y')$$

then

$$I(\Gamma) = \frac{1}{2\pi} \oint_{\Gamma} \frac{X\,dY - Y\,dX}{X^2 + Y^2}$$

$$I(\Gamma) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(-\cos\theta)(\cos\theta) - \sin\theta(\sin\theta)}{1}\, d\theta$$

$$I(\Gamma) = -1$$

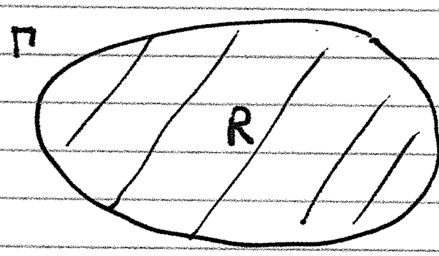
Theorem Let Γ be a simple smooth closed curve which contains no fixed points inside it. Then

$$I(\Gamma) = 0$$

Proof: Since Γ contains no fixed points then $\vec{R}(s) \neq 0$ for any s.

Apply Green's theorem

$$(1) \qquad I(\Gamma) = \frac{1}{2\pi} \oint_{\Gamma} \vec{F} \cdot d\vec{R} = \frac{1}{2\pi} \iint_{R} \frac{\partial F_2}{\partial X} - \frac{\partial F_1}{\partial Y} \, dA$$



Noting

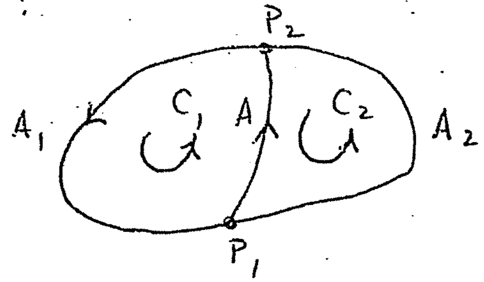
$$\frac{\partial F_1}{\partial X} - \frac{\partial F_2}{\partial Y} = \frac{\partial}{\partial X}\left(\frac{X}{X^2+Y^2}\right) - \frac{\partial}{\partial Y}\left(\frac{-Y}{X^2+Y^2}\right) = \cdots = 0$$

then (1) $\Rightarrow$

$$I(\Gamma) = 0 \qquad \square$$

Defn: A Jordan curve is a piecewise smooth simple (nonintersecting) closed curve.

Theorem: If C is decomposed into two Jordan curves C_1, C_2 with $C = C_1 + C_2$ as:



then

$$I_f(C) = I(C_1) + I(C_2)$$

Proof

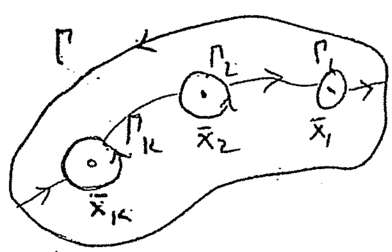
$$\begin{aligned} I_f(C) &= \frac{1}{2\pi}\left(\Delta\phi\big|_{A_1} + \Delta\phi\big|_{A_2} \right) \\ &= \frac{1}{2\pi}\left(\Delta\phi\big|_{A_1} + \Delta\phi\big|_{A} + \Delta\phi\big|_{-A} + \Delta\phi\big|_{A_2} \right) \\ &= \frac{1}{2\pi}\left(\Delta\phi\big|_{C_1} + \Delta\phi\big|_{C_2} \right) \\ &= I_f(C_1) + I_f(C_2). \end{aligned}$$

/

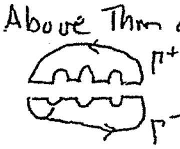
Theorem: If Jordan curve Γ contains a finite number of fixed points (isolated), $\bar{x}_1, \dots \bar{x}_N$ then

$$I(\Gamma) = \sum_{K=1}^{N} I(\Gamma_K)$$

where each Γ_K contains only $\bar{x}_K$:



Pf/ Above Thm and



$I(\Gamma^+) = 0$, $I(\Gamma^-) = 0$