Divergence Theorem in the plane.

Let $\Omega \subset \mathbb{R}^2$ be simply connected with boundary $\partial\Omega$. If $\vec{F} = (P, Q)$ then Green's theorem in the plane is

$$(1) \quad \oint_{\partial\Omega} \vec{F}\cdot d\vec{r} = \oint_{\partial\Omega} P\,dx + Q\,dy = \iint_{\Omega} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

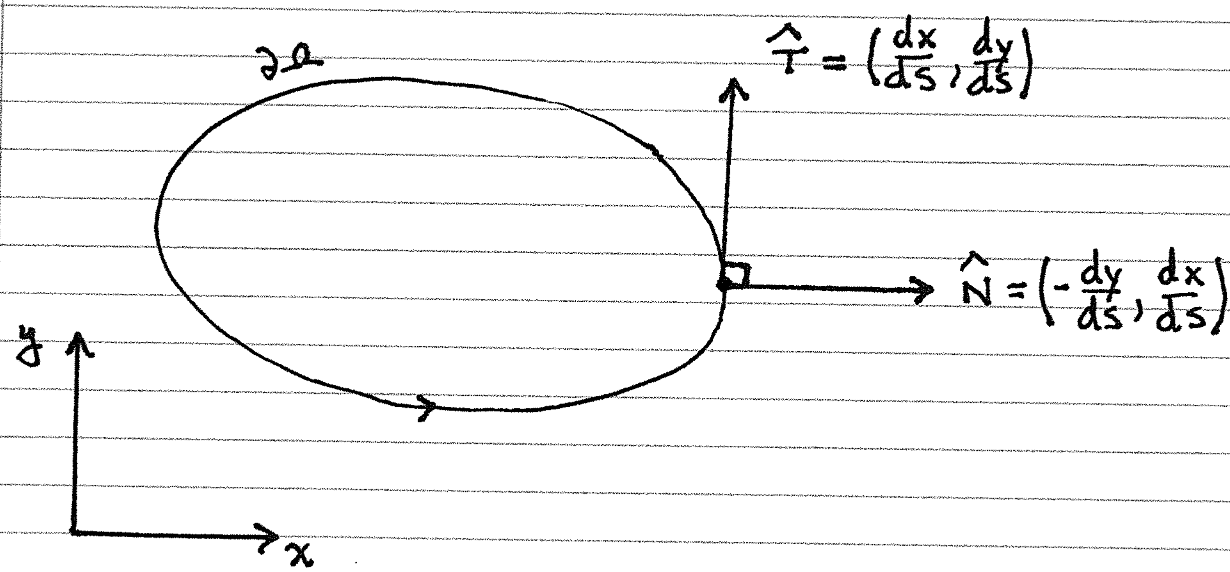
Use the vector field $\vec{f} = (Q, -P)$ in (1) yields:

$$(2) \quad \oint_{\partial\Omega} Q\,dx - P\,dy = \iint_{\Omega} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) dA$$

Given the defn of $\vec{F}$, eqn (2) can be written

$$\boxed{\oint_{\partial\Omega} \vec{F}\cdot\hat{N}\,ds = \iint_{\Omega} \vec{\nabla}\cdot\vec{F}\,dA}$$

where s = arc length parameter and $\hat{N}$ is the outward unit normal to $\partial\Omega$



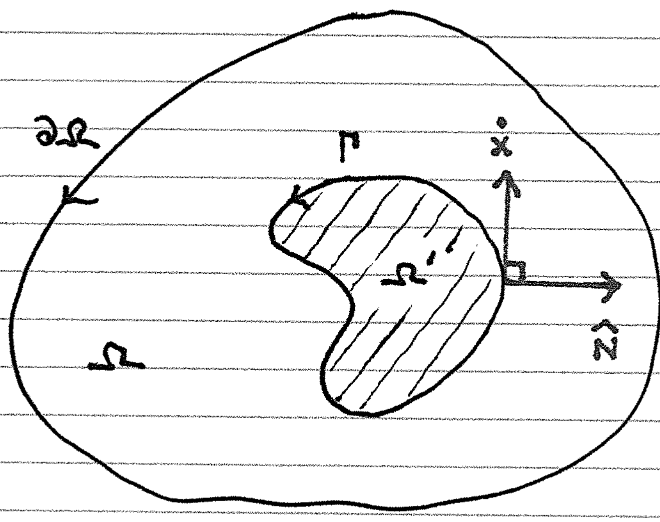
Dulac's Criteria

Let $\dot{x} = \vec{f}(x)$ where $\vec{f} \in C^1(\Omega)$ and Ω is simply conn.
If $\exists\, g(x)$ scalar $g: \mathbb{R}^2 \to \mathbb{R}$ such that

$$|\nabla \cdot (g\vec{f})| \neq 0 \qquad \forall x \in \Omega$$

then $\dot{x} = \vec{f}(x)$ has no closed orbits that lie entirely within Ω.

Pf/ Let Γ be any closed orbit in Ω



Note since Γ is an orbit, $\dot{x}$ is tangent to it.

$$\dot{x} \perp \hat{N}$$

Divergence theorem in the plane

$$\underbrace{\iint_{\Omega'} \nabla \cdot (g\vec{f})\, dA}_{\neq 0 \text{ by assump.}} = \oint_{\Gamma} g \underbrace{(\dot{x} \cdot \hat{N})}_{=0} ds$$

a contradiction. $\square$

Remark: Common choices of g

$$g = 1 \qquad g = x^a y^b \qquad g = e^{ax}, e^{by}$$

Ex Van der Pol Equation

$$\dot{x} = y$$
$$\dot{y} = -x + \mu(1-x^2)y \qquad \mu \neq 0$$

For the choice $g \equiv 1$ one can show

$$\vec{\nabla}\cdot(g\vec{f}) = \mu(1-x^2) \qquad (\text{where } \neq 0\,?)$$

Cannot have a periodic orbit inside rectangle

$$\Omega = \{(x,y) : |x|<1,\ |y|<M\}$$

Ex Competition model has no periodics in quadrant I

$$\dot{N}_1 = r_1 N_1 (1 - N_1/K_1) - b_1 N_1 N_2$$
$$\dot{N}_2 = r_2 N_2 (1 - N_2/K_2) - b_2 N_1 N_2$$

Let Ω = quadrant I and use Dulac Criterion with

$$g = \frac{1}{N_1 N_2}$$

Letting $(x,y) = (N_1, N_2)$ (notational simplicity) after some calculations

$$\vec{\nabla}\cdot(g\vec{f}) = -\frac{(r_1 K_2 x + r_2 K_1 y)}{K_1 K_2\, xy} < 0 \quad \text{on } \Omega$$

Conclude: no physical (pos) periodic solns