Divergence Theorem in the Plane.

Let $\Omega \subset \mathbb{R}^2$ be simply connected with boundary $\partial \Omega$. If $\mathbf{F} = (P, Q)$ then Green's theorem in the plane is

$$\int_{\partial \Omega} \mathbf{F} \cdot d\mathbf{s} = \int_{\Omega} P \, dx + Q \, dy = \iint_{\Omega} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dA$$

Use the vector field $\mathbf{F}^\perp = (Q, -P)$ in (1) yields:

$$\int_{\partial \Omega} Q \, dx - P \, dy = \iint_{\Omega} \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) \, dA$$

Given the defn of $\mathbf{F}^\perp$, eqn (2) can be written

$$\int_{\partial \Omega} \mathbf{F} \cdot \hat{N} \, ds = \iint_{\Omega} \nabla \cdot \mathbf{F} \, dA$$

where $s = \text{arc length parameter}$ and $\hat{N}$ is the outward unit normal to $\partial \Omega$.

\[ \hat{\mathbf{F}} = \left( \frac{dx}{ds}, \frac{dy}{ds} \right) \]

\[ \hat{N} = \left( -\frac{dy}{d\hat{s}} \right) \frac{dx}{d\hat{s}}, \frac{dx}{d\hat{s}} \]
Dulac's Criteria

Let \( \dot{x} = f(x) \) where \( f \in C^1(\mathbb{R}) \) and \( \mathcal{O} \) is simply conn. If there exists a scalar \( g(\mathbb{R}^2 \rightarrow \mathbb{R}) \) such that

\[
\nabla \cdot (g \vec{f}) \neq 0 \quad \forall x \in \mathcal{O}
\]

then \( \dot{x} = f(x) \) has no closed orbits that lie entirely within \( \mathcal{O} \).

**Proof**
Let \( \Gamma \) be any closed orbit in \( \mathcal{O} \).

![Diagram of a closed orbit with a normal vector and divergence theorem](image)

Note since \( \Gamma \) is an orbit, \( \dot{x} \) is tangent to it.

\[
\nabla \cdot (g \vec{f}) \, dA = \oint g(x \cdot \vec{N}) \, ds
\]

\( n' \neq 0 \) by assump., \( \Gamma \cdot n = 0 \)

a contradiction. \( \square \)

**Remark:** Common choices of \( g \)

\[
\begin{align*}
    g &= 1 \\
    g &= x^a y^b \\
    g &= e^{ax} e^{by}
\end{align*}
\]
Ex. **Vander Pol Equation**

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= -x + \mu(1-x^2)y \\
\mu &\neq 0
\end{align*}
\]

For the choice \( q \equiv 1 \) one can show

\[ \nabla \cdot (q F) = \mu(1-x^2) \] (where \( \neq 0 \)?)

Cannot have a periodic orbit inside rectangle

\[ \Omega = \{(x,y): 1x1 < 1, 1y1 < M\} \]

**Ex. Competition model has no periodic in quadrant I**

\[
\begin{align*}
\dot{N}_1 &= r_1 N_1 (1-N_1/K_1) - b_1 N_1 N_2 \\
\dot{N}_2 &= r_2 N_2 (1-N_2/K_2) - b_2 N_1 N_2
\end{align*}
\]

Let \( \Omega = \text{quadrant I} \) and use Dulac Criterion with

\[ q = \frac{1}{N_1 N_2} \]

Letting \((x,y) = (N_1, N_2)\) (notational simplicity)

After some calculations

\[ \nabla \cdot (q F) = - \frac{(r_1 K_2 x + r_2 K_1 y)}{K_1 K_2 x y} < 0 \quad \text{on} \quad \Omega \]

Conclude: no physical (pos) periodic solns